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Conics in the hyperbolic plane

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CONICS IN THE HYPERBOLIC PLANE

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in

Mathematics

by

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Abstract

Let us consider the locus of intersection $L \cap T(L)$, as L runs through the entire pencil of lines through a point P where T is an affine transformation such that $T(P) = Q$. This locus is an affine conic and any affine conic can be produced from this incidence construction. The affine type of the conic (ellipse, parabola, hyperbola) is determined by the invariants of T, the determinant and trace of its linear part. The purpose of this thesis is to obtain a corresponding classification in the hyperbolic plane of conics defined by this construction. We will use the Poincaré disk as our model of the hyperbolic plane, whereby T is a linear fractional transformation that preserves the disk. So that our classification is not model-dependent we will show that the conic produced by T is mapped to an affine conic when dilation about the center of the disk by a hyperbolic factor of 2 is imposed. The resulting intersections of this affine conic with the boundary of the disk coincide with those of the hyperbolic locus, and this hyperbolic conic can be recovered in explicit form by contracting the affine conic. We will find that the type of conic is determined not only by T but also by the distance between points P and Q.

Acknowledgements

I would like to dedicate this project to my brother, Christopher Todd Naeve (1968-2006). This loss was drastic and devastating, but I was able to perservere with extreme focus and concentration. He was always proud of all of my accomplishments and I know he will be proud of this one. Chris was more than just my brother, he was a father, a great companion and a good friend. I love and miss him very much.

I would like to thank Dr. John Sarli, without him, none of this could be possible. I appreciate the time and effort that Dr. Sarli devoted to me, as well as, his vast knowledge and expertise. My only regret is that I had only one opportunity for him to be my instructor during my course of study. The experience of working on my project with Dr. Sarli was the highlight of my Master's program.

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Chapter 1

Introduction

What is a conic? The name comes from the work of the Greek mathematicians Apollonius and Menaechmus who studied the intersection of planes with cones and properties of the resulting ellipses, parabolas, and hyperbolas [BEG99]. We associate these curves with second-degree polynomial equations, that is every second degree equation represents a conic. Most texts define conics in terms of a focus and a directrix. These are important metric properties but do not give any indication of how conics arise synthetically in geometry. The study of conics is well over 2,000 years old and has given rise to some of the most beautiful and striking results in geometry.

Conics arise in a planar geometry as a linear correspondence between a pair of pencils [Ped88]. We can classify and produce affine conics by using the following theorem which we will prove

Theorem 1.1. *If P and Q are points of the affine plane and T is an affine transformation such that $T(P) = Q$, then the locus of intersections $L \cap T(L)$, as L runs through the entire pencil of lines through P, is a conic.*

We will show that the equation of this conic is

$$cx^2 + (d-a)\,xy - by^2 - (a+d)\,ry - cr^2 = 0$$

where T is represented by the matrix:

$$T = \begin{pmatrix} ax & by & r\,(1+a) \\ cx & dy & rc \end{pmatrix}$$

and $P = (-r, 0)$ and $Q = (r, 0)$.

In the affine plane, the invariants of T, the determinant δ and the trace τ of the matrix, will allow us to classify the conic. The goal is to find invariants of T in the hyperbolic plane that function as the trace and determinant in the affine plane.

Poincaré's conformal disk model will be used, where points are those of the open unit disk and lines are circular arcs orthogonal to the unit circle.

Choosing two points P and Q in the hyperbolic plane, a transformation T such that $T(P) = Q$ is now a conformal hyperbolic transformation T takes the pencils of lines (arcs) through P to the pencil through Q. As in the affine plane, the locus of intersections will define a conic. In order to classify the conic determined by T we will use a property of the Poincaré model that relates to the Lobachevsky chord model, specifically, dilation about the center of the disk by a hyperbolic factor of two takes the arc that represents the line to the chord of the disk with the same boundary points. This dilation gives rise to a map from the group of hyperbolic transformations into the Euclidean affine group. This so-called "spinor" map then allows us to place the hyperbolic conic in the same category as its affine counterpart.

The result of this work will show that, while in the affine plane a conic can be classified by three types of cases: an ellipse, a parabola, and a hyperbola, in the hyperbolic plane, conics are classified as one of six types: no intersection, unique intersection, two intersections with either both tangents or a secant, three intersections, and four intersections. Each case is determined by the number of intersections the conic has with the line at infinity which is the unit circle $\mathfrak{C}$.

Chapter 2

The Affine Plane

In geometry, affine transformations of the plane map lines to lines, parallel lines to parallel lines, and preserve ratios of lengths along lines [BEG99]. The affine linear transformation T will be defined by linear equations such that:

$$\begin{aligned} X &= ax + by + p \\ Y &= cx + dy + q \end{aligned}$$

for some constants a, b, c, d, p, q. The transformation T must also be invertible (det $ad - bc \neq 0$) and will be written as a matrix equation,

$$\begin{aligned} T(x, y) &= \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax + by + p \\ cx + dy + q \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}, \text{ and} \end{aligned}$$

$$
\begin{aligned}
T^{-1}(x,y) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right] \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \tfrac{1}{bc-ad} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \tfrac{1}{bc-ad} \begin{pmatrix} dp - bq \\ -cp + aq \end{pmatrix}
\end{aligned}
$$

so,

$$
T^{-1}(x,y) = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & \frac{dp-bq}{bc-ad} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & \frac{-cp+aq}{bc-ad} \\ 0 & 0 & 1 \end{pmatrix}
$$

and

$$
T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc}x & \frac{-b}{ad-bc}y & \frac{dp-bq}{bc-ad} \\ \frac{-c}{ad-bc}x & \frac{a}{ad-bc}y & \frac{-cp+aq}{bc-ad} \\ & 1 & \end{pmatrix}.
$$

The above is the general theory where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the linear part and $\begin{pmatrix} p \\ q \end{pmatrix}$ is the translation part.

For a specific case we will select two points P and Q that will make the distance between them arbitrary. Let $P = (-r, 0)$ and $Q = (r, 0)$. So our transformation T is represented by $T : (-r, 0) \to (r, 0)$

$$T = \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -r \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 1 \end{pmatrix}$$

then solve for p and q

$$\begin{aligned} -ar + b(0) + p &= r \\ p &= r + ar \\ p &= r(1+a) \end{aligned} \qquad \text{and} \qquad \begin{aligned} -cr + d(0) + q &= 0 \\ q &= cr \end{aligned}$$

since $T(P) = Q$. The line L through $P(-r, 0)$ with slope m is represented by the following equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= m(x + r) \\ y &= m(x + r) \end{aligned}$$

and the transformation T is represented by

$$T = \begin{pmatrix} a & b & r(1+a) \\ c & d & cr \\ 0 & 0 & 1 \end{pmatrix}$$

so

$$T^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & \frac{d+ad-bc}{bc-ad} \cdot r \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & \frac{-c}{bc-ad} \cdot r \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc}x & \frac{-b}{ad-bc}y & \frac{d+ad-bc}{bc-ad} \cdot r \\ \frac{-c}{ad-bc}x & \frac{a}{ad-bc}y & \frac{-c}{bc-ad} \cdot r \\ & 1 & \end{pmatrix}$$

must satisfy line L through point $P(-r, 0)$ with slope m:

$$y = m(x+r).$$

This yields,

$$\begin{aligned}
-\tfrac{c}{ad-bc}x + \tfrac{a}{ad-bc}y + \left(\tfrac{-c}{bc-ad}\cdot r\right) &= m\left(\tfrac{d}{ad-bc}x - \tfrac{b}{ad-bc}y + \tfrac{d+ad-bc}{bc-ad}\cdot r + r\right) \\
-\tfrac{c}{ad-bc}x + \tfrac{a}{ad-bc}y - \tfrac{-cr}{bc-ad} &= m\left(\tfrac{d}{ad-bc}x - \tfrac{b}{ad-bc}y + \tfrac{d+ad-bc}{bc-ad}\cdot r + r\right) \\
-\tfrac{c}{ad-bc}x + \tfrac{a}{ad-bc}y - \tfrac{cr}{bc-ad} &= \tfrac{d}{ad-bc}xm - \tfrac{b}{ad-bc}ym + \tfrac{d+ad-bc}{bc-ad}\cdot rm + rm \\
\tfrac{a}{ad-bc}y + \tfrac{b}{ad-bc}ym &= \tfrac{d}{ad-bc}xm + \tfrac{c}{ad-bc}x + \tfrac{d+ad-bc}{bc-ad}\cdot rm + \tfrac{c}{bc-ad}\cdot r + rm \\
\left(\tfrac{a+bm}{ad-bc}\right)y &= \left(\tfrac{dm+c}{ad-bc}\right)x + \tfrac{dm+adm-bcm+bcm-adm}{bc-ad}\cdot r + \tfrac{c}{bc-ad}\cdot r \\
\left(\tfrac{a+bm}{ad-bc}\right)y &= \left(\tfrac{dm+c}{ad-bc}\right)x + \tfrac{dm+c}{bc-ad}r \\
y &= \left(\tfrac{dm+c}{ad-bc}\cdot\tfrac{ad-bc}{a+bm}\right)x + \left(\tfrac{dm+c}{bc-ad}\cdot\tfrac{ad-bc}{a+bm}\right)r \\
y &= \left(\tfrac{dm+c}{a+bm}\right)x - \left(\tfrac{dm+c}{a+bm}\right)r \\
y &= \tfrac{dm+c}{a+bm}(x-r),
\end{aligned}$$

which is the equation of $T(L)$.

Finally, we must determine the intersection of L with $T(L)$ as L runs through the pencil of P and describe this locus of points of intersection.
If we solve for m in the equation for L,

$$\begin{aligned}
L:\ y &= m(x+r) \\
m &= \tfrac{y}{(x+r)}
\end{aligned}$$

and substitute the resulting equation for $T(L)$,

$$
\begin{aligned}
T\left(L\right): \qquad y &= \tfrac{c+dm}{a+bm}\left(x-r\right) \\
\tfrac{y}{\left(x-r\right)} &= \tfrac{c+d\left(\frac{y}{x+r}\right)}{a+b\left(\frac{y}{x+r}\right)} \\
\tfrac{y}{x-r} &= \tfrac{\frac{c}{1}+\frac{dy}{x+r}}{\frac{a}{1}+\frac{by}{x+r}} \\
\tfrac{y}{x-r} &= \tfrac{\frac{c(x+r)+dy}{x+r}}{\frac{a(x+r)+by}{x+r}} \\
\tfrac{y}{x-r} &= \tfrac{c(x+r)+dy}{a(x+r)+by} \\
\tfrac{y}{x-r} &= \tfrac{cx+cr+dy}{ax+ar+by} \\
y\left(ax+ar+by\right) &= \left(x-r\right)\left(cx+cr+dy\right) \\
axy+ary+by^2 &= cx^2+arx+dxy-crx-cr^2-dry \\
cx^2-axy+dxy-by^2-ary-dry-cr^2 &= 0 \\
cx^2+\left(d-a\right)xy-by^2-\left(a+d\right)ry-cr^2 &= 0.
\end{aligned}
$$

This is the equation of the affine conic (ellipse, parabola, or hyperbola) created by the locus of points of intersection [Sar03].

The affine conic is determined by the invariants of T, the determinant and trace of its linear parts. The type of conic in the Cartesian plane is determined by the coefficients of quadratic terms, so the type of conic depends only on the coefficients of x and y in the definition of T. Recall that in the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant $\delta = ad - bc$ and the trace $\tau = a + d$. The type of conic determined by the equation above,

$$\begin{aligned}
\triangle &= \begin{vmatrix} c & \frac{1}{2}(d-a) \\ \frac{1}{2}(d-a) & -b \end{vmatrix} \\
&= -cb - \tfrac{1}{4}(d-a)^2 \\
&= -cb - \tfrac{1}{4}\left(d^2 - 2ad + a^2\right) \\
&= (ad-cb) - \tfrac{1}{4}\left(d^2 + 2ad + a^2\right) \\
&= (ad-cb) - \tfrac{1}{4}(a+d)^2
\end{aligned}$$

will be an ellipse if $\triangle > 0$, a parabola if $\triangle = 0$, or a hyperbola if $\triangle < 0$. Now we can distinguish the type of conic strictly in terms of the determinant δ and trace τ and are able to summarize the results into the following theorem:

Theorem 2.1. *The conic determined by P, Q and transformation T is an ellipse if $\tau^2 < 4\delta$, a parabola if $\tau^2 = 4\delta$, or a hyperbola if $\tau^2 > 4\delta$.*

Let's consider the following three cases: 1) ellipse, 2) parabola, and 3) hyperbola, where point $P(-1,0) \to Q(1,0)$ and a, b, c, d are chosen at random and create the equation of each conic:

Case 1: The Ellipse

Let $a = -3$, $b = -2$, $c = 3$, $d = -2$ and $r = 1$,

$$\text{so} \quad \begin{aligned} p &= r(1+a) \\ &= 1(1+(-3)) \\ &= -2 \end{aligned} \quad \text{and} \quad \begin{aligned} q &= rc \\ &= (1)(3) \\ &= 3 \end{aligned}$$

$$\text{and the matrix } \; T = \begin{pmatrix} -3 & -3 & -2 \\ 3 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will use the determinant δ and trace τ of the matrix to verify that the conic is an ellipse,

$$\begin{aligned} \delta &= ad - bc \\ &= (-3)(-2) - (3)(-2) \\ &= 6 + 6 \\ &= 12 \end{aligned} \qquad \begin{aligned} \tau &= a + d \\ &= (-3) + (-2) \\ &= -5 \end{aligned} \qquad \begin{aligned} \tau^2 &< 4\delta \\ (-5)^2 &< 4(12) \\ 25 &< 48 \end{aligned}$$

so the conic is an ellipse.

The matrix of T^{-1} is

$$
\begin{aligned}
T^{-1} &= \begin{pmatrix} \frac{-2}{12} & \frac{2}{12} \\ \frac{-3}{12} & \frac{-3}{12} \end{pmatrix} \\
&= \begin{pmatrix} \frac{-1}{6} & \frac{1}{6} \\ \frac{-1}{4} & \frac{-1}{4} \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \frac{-1}{4} \end{pmatrix} \\
\text{so } T^{-1} &= \begin{pmatrix} \frac{-1}{6} & \frac{1}{6} & \frac{-5}{6} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} &= \begin{pmatrix} \frac{-1}{6}x & \frac{1}{6}y & \frac{-5}{6} \\ \frac{-1}{4}x & \frac{-1}{4}y & \frac{1}{4} \\ & 1 & \end{pmatrix}
\end{aligned}
$$

and $T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ must satisfy L through $P(-1,0)$ with $y = m(x+1)$ so $T(L)$:

$$
\begin{aligned}
y &= m(x+1) \\
-\tfrac{1}{4}x - \tfrac{1}{4}y + \tfrac{1}{4} &= m\left(-\tfrac{1}{6}x + \tfrac{1}{6}y - \tfrac{5}{6} + 1\right) \\
-\tfrac{1}{4}x - \tfrac{1}{4}y + \tfrac{1}{4} &= -\tfrac{1}{6}mx + \tfrac{1}{6}my + \tfrac{1}{6}m \\
-\tfrac{1}{4}y - \tfrac{1}{6}my &= -\tfrac{1}{6}mx + \tfrac{1}{4}x + \tfrac{1}{6}m - \tfrac{1}{4} \\
\left(-\tfrac{1}{4} - \tfrac{1}{6}m\right) y &= \left(-\tfrac{1}{6}m + \tfrac{1}{4}\right) x + \tfrac{1}{6}m - \tfrac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{-\frac{1}{6}m+\frac{1}{y}}{-\frac{1}{4}-\frac{1}{6}m}x + \frac{\frac{1}{6}-\frac{1}{4}}{-\frac{1}{4}-\frac{1}{6}m} \\
y &= \frac{\frac{-2m+3}{12}}{\frac{-3-2m}{12}}x + \frac{\frac{2m-3}{12}}{\frac{-3-2m}{12}} \\
y &= \frac{-2m+3}{-3-2m}x + \frac{2m-3}{-3-2m} \\
T(L): \ y &= \frac{-2m+3}{-2m-3}(x-1).
\end{aligned}
$$

Solve line L for m, $\left(m = \frac{y}{x-1}\right)$ and substitute in $T(L)$ to find the equation of the conic:

$$
\begin{aligned}
T(L): \quad y &= \frac{-2\left(\frac{y}{x+1}\right)+3}{-2\left(\frac{y}{x+1}\right)-3}(x-1) \\
\frac{y}{x-1} &= \frac{\frac{-2y}{x+1}+3}{\frac{-2y}{x+1}-3} \\
\frac{y}{x-1} &= \frac{-2y+3x+3}{-2y-3x-3}
\end{aligned}
$$

$$
\begin{aligned}
2y^2-3xy-3y &= (x-1)(-2y+3x+3) \\
-2y^2-3xy-3y &= -2xy+3x^2+3x+2y-3x-3 \\
3x^2+xy+2y^2+5y-3 &= 0
\end{aligned}
$$

is the equation of the ellipse.

Graphing the conic gives a visual of the locus of intersection. First, choose equations through point P and transform them by T,

$$\begin{array}{llcll}
L_n: & y = m(x+1) & \rightarrow & T(L_n): & y = \frac{-2m+3}{-2m-3}(x-1) \\
L_1: & y = x+1 & \rightarrow & T(L_1): & y = \frac{-1}{5}x + \frac{1}{5} \\
L_2: & y = 2x+2 & \rightarrow & T(L_2): & y = \frac{1}{7}x - \frac{1}{7} \\
L_3: & y = 3x+3 & \rightarrow & T(L_3): & y = \frac{1}{3}x - \frac{1}{3} \\
L_4: & y = -x-1 & \rightarrow & T(L_4): & y = -5x+5 \\
L_5: & y = -2x-2 & \rightarrow & T(L_5): & y = -7x-7 \\
L_6: & y = -3x-3 & \rightarrow & T(L_6): & y = 3x-3 \\
L_7: & y = \frac{1}{2}x + \frac{1}{2} & \rightarrow & T(L_7): & y = \frac{-1}{2}x + \frac{1}{2} \\
L_8: & y = -\frac{1}{2}x - \frac{1}{2} & \rightarrow & T(L_8): & y = -2x+2 \\
L_9: & y = -6x-6 & \rightarrow & T(L_9): & y = \frac{5}{3}x - \frac{5}{3} \\
L_{10}: & y = -8x-8 & \rightarrow & T(L_{10}): & y = \frac{19}{13}x - \frac{19}{13} \\
L_{11}: & y = -\frac{1}{6}x - \frac{1}{6} & \rightarrow & T(L_{11}): & y = -\frac{19}{17}x + \frac{19}{17} \\
L_{12}: & y = -\frac{2}{3}x - \frac{2}{3} & \rightarrow & T(L_{12}): & y = -\frac{13}{5}x + \frac{13}{5}.
\end{array}$$

Next graph each of the equations L and $T(L)$ respectively and identify their intersections, and the locus of intersections will create an ellipse. See Figure 2.1 and 2.2.

Case 2: The Parabola

Let $a = 4$, $b = -10$, $c = 1$, $d = 12$ and $r = 1$,

$$\text{so} \quad \begin{array}{rcl} p & = & r(1+a) \\ & = & 5 \end{array} \quad \text{and} \quad \begin{array}{rcl} q & = & rc \\ & = & 1 \end{array}$$

$$\text{and the matrix} \quad T = \begin{pmatrix} 4 & -16 & 5 \\ 1 & 12 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The determinant $\delta = 64$ and trace $\tau = 16$ so $\tau^2 = 4\delta$ and the conic is a parabola.

The matrix of T^{-1} is

$$
\begin{aligned}
T^{-1} &= \begin{pmatrix} \frac{12}{64} & \frac{16}{64} \\ \frac{-1}{64} & \frac{4}{64} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{16} & \frac{1}{4} \\ \frac{-1}{64} & \frac{1}{16} \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{19}{16} \\ \frac{-1}{64} \end{pmatrix} \\
\text{so } T^{-1} &= \begin{pmatrix} \frac{3}{16} & \frac{1}{4} & \frac{-19}{16} \\ \frac{-1}{64} & \frac{1}{16} & \frac{1}{64} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} &= \begin{pmatrix} \frac{3}{16}x & \frac{1}{4}y & \frac{-19}{16} \\ \frac{-1}{64}x & \frac{1}{16}y & \frac{1}{64} \\ & 1 & \end{pmatrix}
\end{aligned}
$$

$T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ satisfies L through $P(-1,0)$. So $T(L)$:

$$
\begin{aligned}
y &= m(x+1) \\
-\tfrac{1}{64}x + \tfrac{1}{16}y + \tfrac{1}{64} &= m\left(\tfrac{3}{16}x + \tfrac{1}{4}y - \tfrac{19}{16} + 1\right) \\
-\tfrac{1}{64}x - \tfrac{1}{16}y + \tfrac{1}{64} &= \tfrac{3}{16}mx + \tfrac{1}{4}my - \tfrac{3}{16}m \\
\tfrac{1}{16}y - \tfrac{1}{4}my &= \tfrac{3}{16}mx + \tfrac{1}{64}x - \tfrac{5}{16}m - \tfrac{1}{64} \\
\left(\tfrac{1}{16} - \tfrac{1}{4}m\right)y &= \left(\tfrac{3}{16}m + \tfrac{1}{64}\right)x - \tfrac{3}{16}m - \tfrac{1}{64}
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{\frac{3}{16}m+\frac{1}{64}}{\frac{1}{16}-\frac{1}{4}m}x + \frac{-\frac{3}{16}m-\frac{1}{64}}{\frac{1}{16}-\frac{1}{4}m} \\
y &= \frac{\frac{12m+1}{64}}{\frac{1-4m}{16}}x + \frac{\frac{-12m-1}{64}}{\frac{1-4m}{16}} \\
y &= \frac{12m+1}{4-16m}x - \frac{12m+1}{4-16m} \\
T(L):\ y &= \frac{12m+1}{4-16m}(x-1).
\end{aligned}
$$

Substitute $\frac{y}{x-1}$ for m from line L into $T(L)$ to find the equation of the conic,

$$
\begin{aligned}
T(L): \quad y &= \frac{12\left(\frac{y}{x+1}\right)+1}{4-16\left(\frac{y}{x+1}\right)}(x-1) \\
\frac{y}{x-1} &= \frac{\frac{12y}{x+1}+1}{4-\frac{16y}{x+1}} \\
\frac{y}{x-1} &= \frac{12y+x+1}{4x+4-16y} \\
y(4x+4-16y) &= (x-1)(12y+x+1) \\
-2y^2-3xy-3y &= -2xy+3x^2+3x+2y-3x-3 \\
4xy+4y-16y^2 &= 12xy+x^2+x-12y-x-1 \\
x^2+8xy+16y^2-16y-1 &= 0
\end{aligned}
$$

is the equation of the parabola.

Graphing the conic in this case yields the following equations,

$$\begin{array}{llcll}
L_n: & y = m(x+1) & \to & T(L_n): & y = \frac{12m+1}{4-16m}(x-1) \\
L_1: & y = x+1 & \to & T(L_1): & y = -\frac{13}{12}x + \frac{13}{12} \\
L_2: & y = 2x+2 & \to & T(L_2): & y = -\frac{25}{28}x + \frac{25}{28} \\
L_3: & y = 3x+3 & \to & T(L_3): & y = -\frac{37}{44}x + \frac{37}{44} \\
L_4: & y = 4x+4 & \to & T(L_4): & y = -\frac{49}{60}x + \frac{49}{60} \\
L_5: & y = -x-1 & \to & T(L_5): & y = -\frac{11}{20}x + \frac{11}{20} \\
L_6: & y = -2x-2 & \to & T(L_6): & y = -\frac{23}{36}x + \frac{23}{36} \\
L_7: & y = -3x-3 & \to & T(L_7): & y = -\frac{35}{52}x + \frac{35}{52} \\
L_8: & y = -\frac{1}{2}x - \frac{1}{2} & \to & T(L_8): & y = -\frac{5}{12}x + \frac{5}{12} \\
L_9: & y = \frac{1}{2}x + \frac{1}{2} & \to & T(L_9): & y = -\frac{7}{4}x + \frac{7}{4} \\
L_{10}: & y = -\frac{1}{6}x - \frac{1}{6} & \to & T(L_{10}): & y = -\frac{3}{20}x + \frac{3}{20} \\
L_{11}: & y = -\frac{1}{8}x - \frac{1}{8} & \to & T(L_{11}): & y = -\frac{1}{12}x + \frac{1}{12} \\
L_{12}: & y = -\frac{1}{12}x - \frac{1}{12} & \to & T(L_{12}): & y = 0.
\end{array}$$

The locus of intersections will create a parabola. See Figure 2.3 and 2.4.

Case 3: The Hyperbola

Let $a = 3$, $b = -2$, $c = 3$, $d = -5$ and $r = 1$ so $p = 4$ and $q = 3$

$$\text{the matrix} \quad T = \begin{pmatrix} 3 & -2 & 4 \\ 3 & -5 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

The determinant $\delta = -9$ and trace $\tau = -2$ so $\tau^2 > 4\delta$ and the conic is a hyperbola.

The matrix of T^{-1} is

$$\begin{aligned} T^{-1} &= \begin{pmatrix} \frac{5}{9} & -\frac{2}{9} \\ \frac{3}{9} & -\frac{3}{9} \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{9} & -\frac{2}{9} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{14}{19} \\ \frac{1}{3} \end{pmatrix} \end{aligned}$$

$$\text{so } T^{-1} = \begin{pmatrix} \frac{5}{9} & -\frac{2}{9} & -\frac{14}{9} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$T^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{9}x & \frac{-2}{9}y & -\frac{14}{9} \\ \frac{1}{3}x & -\frac{1}{3}y & -\frac{1}{3} \\ & 1 & \end{pmatrix}$$

$T^{-1}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ satisfies L through $P(-1,0)$. So $T(L)$:

$$\begin{aligned}
y &= m(x+1) \\
\tfrac{1}{3}x - \tfrac{1}{3}y - \tfrac{1}{3} &= m\left(\tfrac{5}{9}x - \tfrac{2}{9}y - \tfrac{14}{19} + 1\right) \\
\tfrac{1}{3}x - \tfrac{1}{3}y - \tfrac{1}{3} &= \tfrac{5}{9}mx - \tfrac{2}{9}my - \tfrac{5}{9}m \\
-\tfrac{1}{3}y + \tfrac{2}{9}my &= \tfrac{5}{9}mx - \tfrac{1}{3}x - \tfrac{5}{9}m + \tfrac{1}{3} \\
\left(-\tfrac{1}{3} + \tfrac{2}{9}m\right)y &= \left(\tfrac{5}{9}m - \tfrac{1}{3}\right)x - \tfrac{3-5m}{9} \\
y &= \frac{\frac{5}{9}m - \frac{1}{3}}{-\frac{1}{3} + \frac{2}{9}m}x + \frac{\frac{3-5m}{9}}{-\frac{1}{3} + \frac{2}{9}m} \\
y &= \frac{\frac{5m-3}{9}}{\frac{2m-3}{9}}x + \frac{\frac{3-5m}{9}}{\frac{2m-3}{9}} \\
y &= \frac{5m-3}{2m-3}x - \frac{5m-3}{2m-3} \\
T(L): \quad y &= \frac{5m-3}{2m-3}(x-1)
\end{aligned}$$

substitute $\frac{y}{x-1}$ for m from line L into $T(L)$ to find the equation of the conic,

$$
\begin{aligned}
T(L): \quad y &= \frac{5\left(\frac{y}{x+1}\right)-3}{2\left(\frac{y}{x+1}\right)-3}(x-1) \\
\frac{y}{x-1} &= \frac{\frac{5y}{x+1}-3}{\frac{2y}{x+1}-3} \\
\frac{y}{x-1} &= \frac{5y-3x-3}{2y-3x-3} \\
y(2y-3x-3) &= (x-1)(5y-3x-3) \\
2y^2-3xy-3y &= 5xy-3x^2-3x-5y+3x+3 \\
3x^2-8xy+2y^2+2y-3 &= 0
\end{aligned}
$$

is the equation of the hyperbola. Graphing the conic in this case yields the following equations,

$$
\begin{array}{llcll}
L_n: & y = m(x+1) & \to & T(L_n): & y = \frac{5m-3}{2m-3}(x-1) \\
L_1: & y = x+1 & \to & T(L_1): & y = 2x-2 \\
L_2: & y = 2x+2 & \to & T(L_2): & y = 7x-7 \\
L_3: & y = 3x+3 & \to & T(L_3): & y = 4x-4 \\
L_4: & y = 4x+4 & \to & T(L_4): & y = \frac{17}{5}x - \frac{17}{5} \\
L_5: & y = -x-1 & \to & T(L_5): & y = \frac{8}{5}x - \frac{8}{5} \\
L_6: & y = -2x-2 & \to & T(L_6): & y = \frac{13}{7}x - \frac{13}{7} \\
L_7: & y = -3x-3 & \to & T(L_7): & y = 2x-2 \\
L_8: & y = -4x-4 & \to & T(L_8): & y = \frac{23}{11}x - \frac{23}{11} \\
L_9: & y = -\frac{1}{2}x - \frac{1}{2} & \to & T(L_9): & y = \frac{11}{8}x - \frac{11}{8} \\
L_{10}: & y = \frac{1}{2}x + \frac{1}{2} & \to & T(L_{10}): & y = \frac{1}{4}x - \frac{1}{4} \\
L_{11}: & y = \frac{1}{4}x + \frac{1}{4} & \to & T(L_{11}): & y = \frac{7}{10}x - \frac{7}{10} \\
L_{12}: & y = -\frac{1}{4}x - \frac{1}{4} & \to & T(L_{12}): & y = \frac{17}{14}x - \frac{17}{14} \\
L_{13}: & y = \frac{9}{16}x + \frac{9}{16} & \to & T(L_{13}): & y = \frac{1}{10}x - \frac{1}{10} \\
L_{14}: & y = \frac{15}{32}x + \frac{15}{32} & \to & T(L_{14}): & y = \frac{7}{22}x - \frac{7}{22}.
\end{array}
$$

The locus of intersections will create a hyperbola. See Figure 2.5 and 2.6.

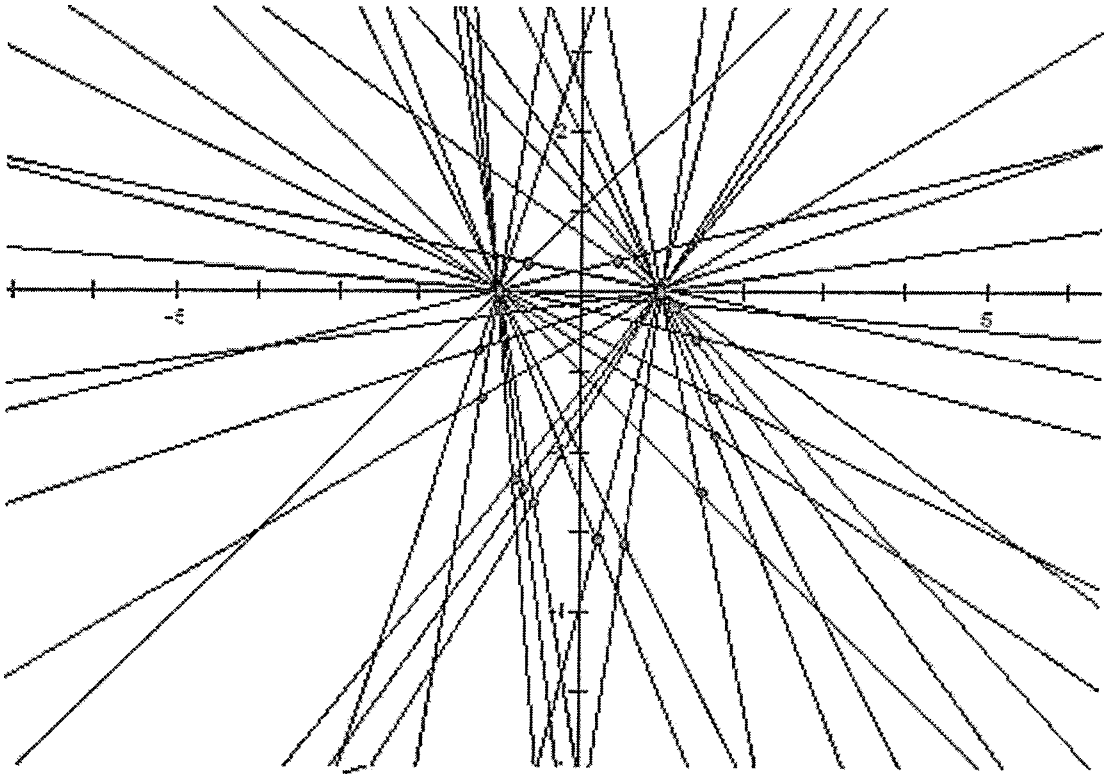

Figure 2.1: The intersections of L and $T(L)$ that create an ellipse.

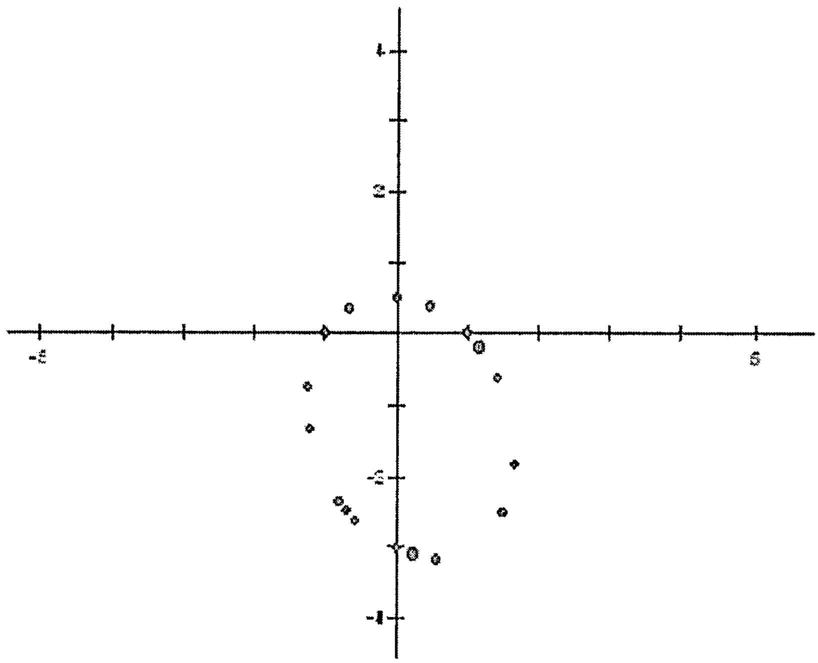

Figure 2.2: The ellipse.

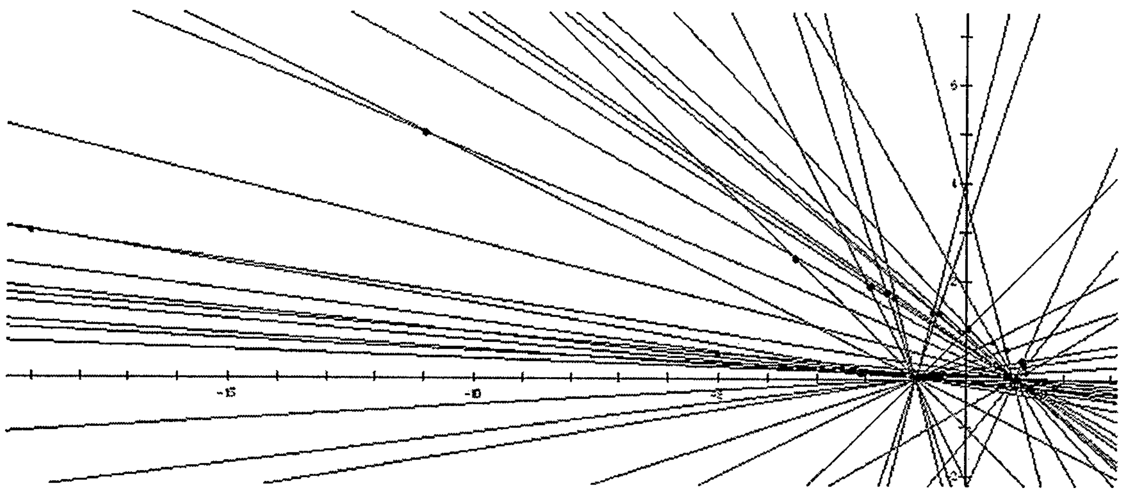

Figure 2.3: The intersections of L and $T\,(L)$ that create the parabola.

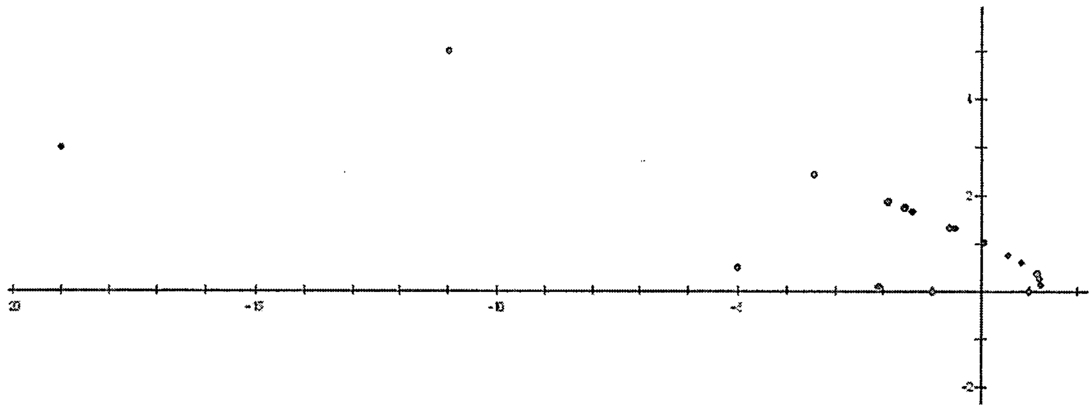

Figure 2.4: The parabola.

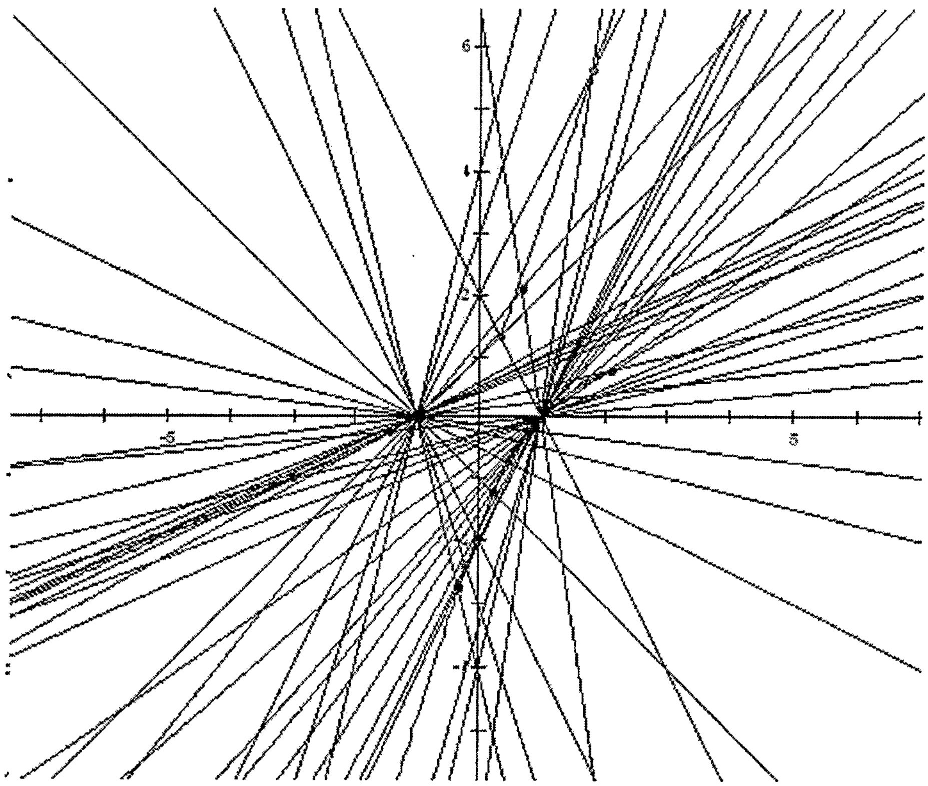

Figure 2.5: The intersections of L and $T\,(L)$ that create the hyperbola.

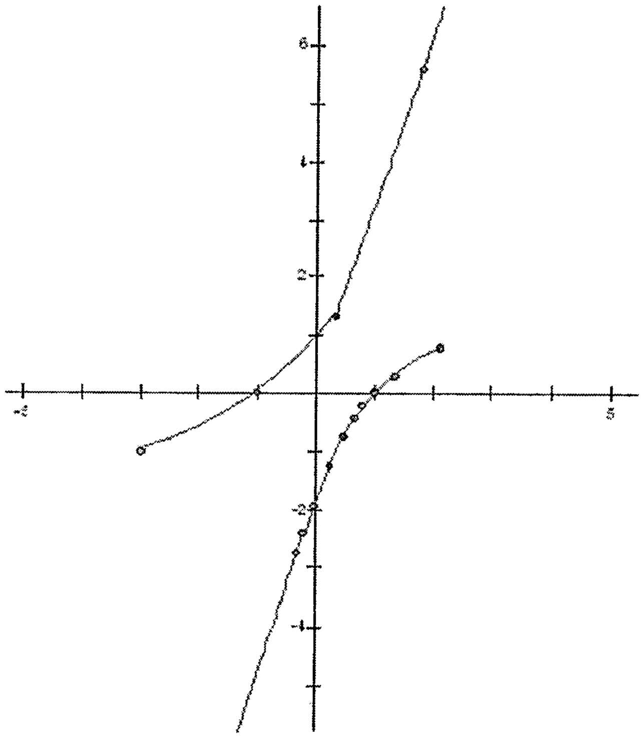

Figure 2.6: The hyperbola.

Chapter 3

Hyperbolic Plane

Non-Euclidean geometry was one of the most momentous mathematical discoveries of the 19th century. Hyperbolic geometry is a non-Euclidean geometry which obeys all the Euclidean postulates except the parallel postulate [Ped88]. In Hyperbolic geometry, the Poincaré disk model, also called the conformal disk model, is a model in which points of the geometry are in the open unit disk $\mathfrak{D}$. The lines of the geometry are segments of circles (arcs) contained in the disk orthogonal to the boundary circle $\mathfrak{C}$ and this also includes diameters that are orthogonal to the disk.

Choosing two points P and Q in the hyperbolic plane, a transformation T such that $T(P) = Q$ is now a conformal hyperbolic transformation T which takes the pencil of circles through P to the pencil through Q. Every direct non-Euclidean transformation can be described as a Möbius transformation of the form:

$$T(z) = \frac{az + b}{\overline{b}z + \overline{a}}$$

where $\frac{|b|}{|a|} < 1$.
The matrix associated with the transformation is

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}.$$

This transformation leaves orientations unchanged and is called a direct non-Euclidean transformation [BEG99]. A direct transformation is called: a rotation if T has a fixed point in $\mathfrak{D}$, a translation if T has no fixed point in $\mathfrak{D}$ but two distinct points on $\mathfrak{C}$, and a limit rotation if T has no fixed point in $\mathfrak{D}$ but a unique fixed point on $\mathfrak{C}$.

The direct transformation above results in the following theorem:

Theorem 3.1. *A direct non-Euclidean transformation T can be written in the form $T(z) = K\frac{z-m}{1-\overline{m}z}$ where K and m are complex numbers with $|K| = 1$ and $|m| < 1$.*

This is the canonical form of a direct non-Euclidean transformation [BEG99]. Note that the transformation T with $m = -R$ will map the point $-R$ of $\mathfrak{D}$ to the origin. By two applications of the theorem, we can obtain the general form of a direct non-Euclidean transformation that maps $-R$ to R in $\mathfrak{D}$. Let $T_1(z)$ be the transformation that takes $-R$ to the origin and $T_2(z)$ be the transformation that takes R to the origin, so

$$T_1(z) = K_1\frac{z+R}{1-Rz} \quad \text{and} \quad T_2(z) = K_2\frac{z-R}{1+Rz}$$

the matrices that are associated with the above transformations are

$$A_1 = \begin{pmatrix} K_1 & RK_1 \\ -R & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} K_2 & -RK_2 \\ R & 1 \end{pmatrix}.$$

The inverse of A_2,

$$A_2^{-1} = \begin{pmatrix} 1 & R \\ -RK_2 & K_2 \end{pmatrix}$$

so

$$\begin{aligned} A_2^{-1}A_1 &= \begin{pmatrix} 1 & R \\ -RK_2 & K_2 \end{pmatrix}\begin{pmatrix} K_1 & RK_1 \\ -R & 1 \end{pmatrix} \\ \text{and let } K_1 = K - 2 &= \begin{pmatrix} K - R^2 & R(K+1) \\ -R(K+1) & -R^2K+1 \end{pmatrix}. \end{aligned}$$

Hence any direct transformation that maps $-R$ to R may be written in the form

$$T(z) = \frac{(K - R^2)z + (RK+1)}{-R(K+1)z + (-R^2K+1)}, \text{ where } |K| = 1.$$

Thus, in correspondence with $|K| = 1$, there are an infinite number of transformations that take $-R$ to R. First, we will dispense of the trivial case. Let $T_1 = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ and $T_2 = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix}$ where $T_2(0) = R$.

$$\begin{aligned} T_1(-R) &= \tfrac{-aR+b}{-\bar{b}R+\bar{a}} &&= R \quad \text{and} \quad T_2 = \tfrac{d}{\bar{c}} = R \\ & -aR+b &&= 0 \qquad\qquad\qquad\quad d = R\bar{c} \\ & b &&= aR \end{aligned}$$

$$T_1 = \begin{pmatrix} a & aR \\ \bar{a}R & \bar{a} \end{pmatrix} \qquad T_2 = \begin{pmatrix} c & R\bar{c} \\ Rc & \bar{c} \end{pmatrix}$$

$$\begin{aligned} T_2T_1 &= \begin{pmatrix} c & R\bar{c} \\ Rc & \bar{c} \end{pmatrix} \begin{pmatrix} a & Ra \\ R\bar{a} & \bar{a} \end{pmatrix} \\ &= \begin{pmatrix} ac + R^2\bar{a}\bar{c} & R(ac + \bar{a}\bar{c}) \\ R(ac + \bar{a}\bar{c}) & R^2ac + \bar{a}\bar{c} \end{pmatrix} \\ &= \begin{pmatrix} ac + R^2\bar{a}\bar{c} & 0 \\ 0 & R^2ac + \bar{a}\bar{c} \end{pmatrix} \\ &= \begin{pmatrix} i(1-R^2) & 0 \\ 0 & i(R^2-1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

which takes $-R$ to R. This is nothing more than multiplication by -1 which takes the real number line and maps it to itself. This case will result in a degenerate conic, which is the real axis.

Next, we will find a single parameter t where every value of t will generate a different conic. Let $R = \tanh p$ and $p = \frac{d}{2}$ where $\frac{d}{2}$ is the distance from the origin. If we assume that $ac \neq \alpha i$, then

$$T_2T_1 = \begin{pmatrix} \frac{ac+R^2\bar{a}\bar{c}}{R(ac+\bar{a}\bar{c})} & 1 \\ 1 & \frac{\bar{a}\bar{c}+R^2ac}{R(ac+\bar{a}\bar{c})} \end{pmatrix}$$

and since $R = \tanh \frac{d}{2}$,

$$\begin{aligned} \frac{ac+R^2\bar{a}\bar{c}}{2R} &= \frac{ac+\tanh^2 \frac{d}{2}\bar{a}\bar{c}}{2\tanh \frac{d}{2}} \\ &= \frac{1+\tanh^2 \frac{d}{2}}{2\tanh \frac{d}{2}} + i\left[\frac{\left(1-\tanh^2 \frac{d}{2}\right)\cdot ac}{2\tanh \frac{d}{2}\cdot ac}\right]. \end{aligned}$$

But $\frac{1+\tanh^2 \frac{d}{2}}{2\tanh \frac{d}{2}} = \coth d$, and if we let our parameter $t = \frac{\left(1-\tanh^2 \frac{d}{2}\right)\cdot ac}{2\tanh \frac{d}{2}\cdot ac}$, then after substituting we get $\coth d+it$. This is the desired transformation $T = \begin{pmatrix} \coth d + it & 1 \\ 1 & \coth d - it \end{pmatrix}$ that takes $-R$ to R.

Given the above transformation $T = \begin{pmatrix} \coth d + it & 1 \\ 1 & \coth d - it \end{pmatrix}$, the conic that is produced by the correspondence between $P(-R)$ and $Q(R)$ is determined by t, so the conic is determined by $\coth d + it$ where p and t will determine whether T is a rotation, translation, or a limit rotation. In each case we must solve for the fixed points of T:

$$\begin{aligned} T(z) = \frac{(\coth d+it)z+1}{z+(\coth d-it)} &= z \\ z^2 + (\coth d - it)\, z &= (\coth d + it)\, z + 1 \\ z^2 - (zit) - 1 &= 0 \end{aligned}$$

$$\text{then } z = \frac{2it \pm \sqrt{-4t^2+4}}{2}$$

$$= it \pm \sqrt{1-t^2}$$

so $z_1 = it + \sqrt{1-t^2}$ and $z_2 = it - \sqrt{1-t^2}$. When $|t| > 1$ we get a rotation, if $|t| < 1$ a translation, and when $t = \pm 1$ we will get a limit rotation [Sch79].

Lines in the hyperbolic plane are circular arcs orthogonal to the unit circle. If we let $H = \begin{pmatrix} a & \beta \\ \bar{\beta} & d \end{pmatrix}$ then the determinant of $H = ad - |\bar{\beta}|^2$ and if the determinant $H < 0$, then H will represent a circle. For example, let $H = \begin{pmatrix} 1 & \coth d + is \\ \coth d - is & 1 \end{pmatrix}$ then

$$\begin{aligned}
\begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} 1 & \coth d + is \\ \coth d - is & 1 \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} &= 0 \\
\begin{pmatrix} z + \coth d - is & z(\coth d + is) + 1 \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} &= 0 \\
z\bar{z} + (\coth d - is)\,\bar{z} + (\coth d + is)\,z + 1 &= 0 \\
|z|^2 + 2\left[z(\coth d + is)\right] + 0 &= 0.
\end{aligned}$$

Let $z = x + iy$, then from the above equation:

$$\begin{aligned}
x^2 + y^2 + 2(x \coth d - ys) + 1 &= 0 \\
x^2 + (2\coth d)\,x + y^2 - 2sy &= -1 \\
(x + \coth d)^2 + (y - s)^2 &= -1 + \coth^2 d + s^2.
\end{aligned}$$

The center of the circle is $(\coth d, s)$ and the radius of the circle is $\sqrt{\coth^2 d + s^2 - 1}$. So any line in the Poincaré plane is in the form of $\begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix}$, where $|\beta| > 1$ and $\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$ if that line passes through the origin.

A line L, that runs through $-R$ can be represented by the following Hermitian matrix,

$$H = \begin{pmatrix} 1 & \coth d + is \\ \coth d - is & 1 \end{pmatrix} \text{ for some } s.$$

If we let $s = \frac{1-R^2}{2R} \cot\theta$, then

$$H = \begin{pmatrix} 2R\sin\theta & \left(1+R^2\right)\sin\theta + i\left(1-R^2\right)\cos\theta \\ \left(1+R^2\right)\sin\theta - i\left(1-R^2\right)\cos\theta & 2R\sin\theta \end{pmatrix}, 0 \leq \theta \leq \pi$$

and

$$\begin{aligned} \begin{pmatrix} -R & 1 \end{pmatrix} H \begin{pmatrix} -R \\ 1 \end{pmatrix} &= 0 \\ R^2\left(2R\sin\theta\right) - 2R\left(1+R^2\right)\sin\theta + 2R\sin\theta &= 0 \\ 2R\sin\theta\left(R^2 - 2R + 1\right) &= 0 \\ R^2 - 2R + 1 &= 0. \end{aligned}$$

But $R = \tanh\frac{d}{2}$, so by substitution in the above equation, is equal to the following: $\tanh^2\frac{d}{2} - 2\tanh\frac{d}{2} + 1 = 0$. The lines through R are represented by

$H = \begin{pmatrix} 1 & -\coth d + is \\ -\coth d - is & 1 \end{pmatrix}$, for some s. If we use the same $s = \frac{1-R^2}{2R}\cot\theta$ as above, then

$$H = \begin{pmatrix} 2R\sin\theta & -\left(1+R^2\right)\sin\theta - i\left(1-R^2\right)\cos\theta \\ -\left(1+R^2\right)\sin\theta - i\left(1-R^2\right)\cos\theta & 2R\sin\theta \end{pmatrix},$$

and then by using the previous calculation from above $\begin{pmatrix} R & 1 \end{pmatrix} H \begin{pmatrix} R \\ 1 \end{pmatrix} = 0$ will result in the same equation, $\tanh^2\frac{d}{2} - 2\tanh\frac{d}{2} + 1 = 0$.

Recall that inversive geometry considers lines and circles in the plane to be the same objects by adjoining the point ∞. The inversive transformations form a group T consisting of the linear fractional transformations (LFT) and compositions of LFT's with complex conjugation [Sch79]. We can now compute the image of a circle C under the action of inverse transformations. If H is the Hermitian matrix of the circle C we have $ZHZ^* = 0$, where $Z = \begin{pmatrix} z & 1 \end{pmatrix}$. We want a matrix $\tilde{H}$ such that $T(z)$ is on the line represented by $\tilde{H}$, so $\begin{pmatrix} T(z) & 1 \end{pmatrix} \tilde{H} \begin{pmatrix} T(\bar{z}) \\ 1 \end{pmatrix} = 0$. If T^+ represents the transpose of T, then this implies that $ZT^+ H \bar{T} \bar{Z}^+ = 0$ and so $(T^+)^{-1} H (\bar{T})^{-1}$ is the Hermitian matrix $\tilde{H}$ of the circle $T(C)$. Note that $(T^+)^{-1} = T_{co}$, the cofactor matrix of T, and $(\bar{T})^{-1} = T^*_{co}$ (the cofactor conjugate). This proves the following:

Theorem 3.2. *If H is the Hermitian matrix of the circle C and T is the matrix of a LFT, then $T_{co} H T^*_{co}$ is the Hermitian matrix of $T(C)$.*

The matrix $T_{co} H T^*_{co}$ is called the spin conjugate of H by T. This spin action is the principal method of transforming circles in $\hat{C}$.

If H is in the pencil through P, then the image $T(H)$ is as follows:

Let

$$
\begin{aligned}
T &= \begin{bmatrix} \coth d + it & 1 \\ 1 & \coth d - it \end{bmatrix}, \\
H &= \begin{pmatrix} 1 & \coth d + is \\ \coth d - is & 1 \end{pmatrix}, \\
T_{co} &= \begin{bmatrix} \coth d - it & -1 \\ -1 & \coth d + it \end{bmatrix} \text{ and} \\
T^*_{co} &= \begin{bmatrix} \coth d + it & -1 \\ -1 & \coth d - it \end{bmatrix}
\end{aligned}
$$

so $T(H) = T_{co} H T^*_{co}$

$$
\begin{bmatrix} \coth d - it & -1 \\ -1 & \coth d + it \end{bmatrix} \begin{pmatrix} 1 & \coth d + is \\ \coth d - is & 1 \end{pmatrix} \begin{bmatrix} \coth d + it & -1 \\ -1 & \coth d - it \end{bmatrix}.
$$

We want to change the parameter t similarly to the change of s. These simple parameter changes create all the hyperbolic rotations about $-R$ to R. So let $s = \frac{1-R^2}{2R}\cot\theta$ and $t = \frac{R^2-1}{2R}\cot\frac{\phi}{2}$, then

$$T = \begin{bmatrix} \coth d + it & 1 \\ 1 & \coth d - it \end{bmatrix}$$

becomes

$$T = \begin{bmatrix} \left(1+R^2\right)\sin\frac{\phi}{2} - i\left(1-R^2\right)\cos\frac{\phi}{2} & 2R\sin\frac{\phi}{2} \\ 2R\sin\frac{\phi}{2} & \left(1+R^2\right)\sin\frac{\phi}{2} + i\left(1-R^2\right)\cos\frac{\phi}{2} \end{bmatrix},$$

where $0 \le \phi \le \pi$, and $T(H) = T_{co}HT_{co}^*$, where

$$T_{co} = \begin{bmatrix} \left(R^2+1\right)\left(\sin\frac{\phi}{2}\right) + i\left(1-R^2\right)\left(\cos\frac{\phi}{2}\right) & -2R\sin\frac{\phi}{2} \\ -2R\sin\frac{\phi}{2} & \left(R^2+1\right)\left(\sin\frac{\phi}{2}\right) - i\left(1-R^2\right)\left(\cos\frac{\phi}{2}\right) \end{bmatrix}$$

$$H = \begin{pmatrix} 2R\sin\theta & \left(1+R^2\right)\sin\theta + i\left(1-R^2\right)\cos\theta \\ \left(1+R^2\right)\sin\theta - i\left(1-R^2\right)\cos\theta & 2R\sin\theta \end{pmatrix},$$

$$T_{co}^* = \begin{bmatrix} \left(R^2+1\right)\left(\sin\frac{\phi}{2}\right) - i\left(1-R^2\right)\left(\cos\frac{\phi}{2}\right) & -2R\sin\frac{\phi}{2} \\ -2R\sin\frac{\phi}{2} & \left(R^2+1\right)\left(\sin\frac{\phi}{2}\right) - i\left(1-R^2\right)\left(\cos\frac{\phi}{2}\right) \end{bmatrix}$$

so, $T(H) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where

$$\begin{aligned} a &= 2R\sin(\theta+\phi), \\ b &= -\left(R^2+1\right)\sin(\theta+\phi) + i\left(R^2-1\right)\cos(\theta+\phi), \\ c &= -\left(R^2+1\right)\sin(\theta+\phi) - i\left(R^2-1\right)\cos(\theta+\phi), \text{ and} \\ d &= 2R\sin(\theta+\phi). \end{aligned}$$

Chapter 4

Spinor Correspondence

We now have representation of every line through $-R = -\tanh\frac{d}{2}$ and a description of every direct isometry that takes $-R$ to R. We can now find the conic determined by the locus $L \cap T(L)$.

However this locus will be complicated to find so we will simplify our work by finding instead the locus of intersections of chords obtained by dilating these lines by a hyperbolic factor of 2. Thus we reduce our work to a corresponding affine conic obtained as in the previous chapter.

The radial dilation of the arcs through $-R$ transforms the arcs into chords that share the endpoints on the boundary circle $\mathfrak{C}$. Previoiusly we let $R = \tanh\frac{d}{2}$ and $d(-R, R) = d$. When we dilate by the hyperbolic factor of 2 we get $\frac{2R}{1+R^2} = \frac{2\tanh\frac{d}{2}}{1+\tanh^2\frac{d}{2}} = \tanh d$. This will dilate the entire pencil of lines through $-R$ to chords that go through the point $\frac{-2R}{1+R^2}$ and can be written as the following theorem.

Theorem 4.1. *If the Hermitian matrix* $\begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix}$ *represents a Poincaré line for some* β*, with* $|\beta| > 1$*, then the corresponding dilated chord with the same endpoints is represented by* $\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 2 \end{pmatrix}$.

Proof:

Let $Re^{i\theta}$ be any point on $\begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix}$. We want to show that $\frac{2R}{1+R^2}e^{i\theta}$ is on $\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 2 \end{pmatrix}$.

$$\begin{aligned}
\Longrightarrow (Re^{i\theta}) \begin{pmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{pmatrix} \begin{pmatrix} Re^{-i\theta} \\ 0 \end{pmatrix} &= 0 \\
\begin{pmatrix} Re^{i\theta} + \bar{\beta}, & \beta Re^{i\theta} + 1 \end{pmatrix} \begin{pmatrix} Re^{-i\theta} \\ 1 \end{pmatrix} &= 0 \\
(Re^{i\theta} + \bar{\beta})\, Re^{-i\theta} + \beta Re^{i\theta} + 1 &= 0 \\
R^2 + \bar{\beta} Re^{-i\theta} + \beta Re^{i\theta} + 1 &= 0 \\
\bar{\beta}\tfrac{R}{1+r^2}e^{-i\theta} + \beta\tfrac{R}{1+R^2}e^{i\theta} + \tfrac{R^2+1}{R^2+1} &= 0.
\end{aligned}$$

Then multiply by 2,

$$\tfrac{2R}{1+R^2}\bar{\beta}e^{-i\theta} + \tfrac{2R}{1+R^2}\beta e^{i\theta} + 2 = 0.$$

We need to check,

$$\begin{aligned}
\begin{pmatrix} \frac{2R}{1+R^2}e^{i\theta} & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 2 \end{pmatrix} \begin{pmatrix} \frac{2R}{1+R^2}e^{i\theta} \\ 1 \end{pmatrix} &= 0 \\
\begin{pmatrix} \bar{\beta} & \frac{2R}{1+R^2}\beta e^{i\theta} + 2 \end{pmatrix} \begin{pmatrix} \frac{2R}{1+R^2}{}^{i\theta} \\ 1 \end{pmatrix} &= 0 \\
\tfrac{2R}{1+R^2}\bar{\beta}e^{-i\theta} + \tfrac{2R}{1+R^2}\beta e^{i\theta} + 2 &= 0,
\end{aligned}$$

so indeed $\frac{2R}{1+R^2}e^{i\theta}$ is on $\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 2 \end{pmatrix}$.

$\square$

Recall that if the straight line L through $(-R, 0)$ is given by $y = m(x+R)$ and A is an affine transformation $(-R, 0)$ to $(R, 0)$, with linear part $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the line $A(L)$ is given by

$$y = \tfrac{c+dm}{a+bm}(x-R).$$

If the hyperbolic line through $-R = -\tanh\frac{d}{2}$ is represented by a Hermitan matrix

$$H = \begin{pmatrix} 2R\sin\theta & \left(1+R^2\right)\sin\theta + i\left(1-R^2\right)\cos\theta \\ \left(1+R^2\right)\sin\theta - i\left(1-R^2\right)\cos\theta & 2R\sin\theta \end{pmatrix} \text{ then H}$$

dilates to the chord through $-\frac{2R}{1+R^2}$ with slope $m = \frac{1+R^2}{1-R^2}\tan\theta$ represented by

$$\begin{pmatrix} 0 & \frac{\left(1+R^2\right)}{2R} + i\frac{\left(1-R^2\right)}{2R}\cot\theta \\ \frac{\left(1+R^2\right)}{2R} - i\frac{\left(1-R^2\right)}{2R}\cot\theta & 2 \end{pmatrix}.$$

Then $T(H) = \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix}$ where,

$$\begin{aligned} a &= 2R\sin(\theta+\phi), \\ b &= -\left(R^2+1\right)\sin(\theta+\phi) + i\left(R^2-1\right)\cos(\theta+\phi), \text{ and} \\ \bar{b} &= -\left(R^2+1\right)\sin(\theta+\phi) - i\left(R^2-1\right)\cos(\theta+\phi) \end{aligned}$$

dilates to a chord through $\frac{2R}{1+R^2}$ with slope $M = \frac{1+R^2}{1-R^2}\tan(\theta+\phi)$.

We can now find an affine transformation A that takes chords through $-R$ to chords through R in correspondence with the action of T on the hyperbolic lines, by determining constants a, b, c, d. Since $M = \frac{c+dm}{a+bm}$, then

$$\tfrac{1+R^2}{1-R^2}\tan(\theta+\phi) = \frac{c+d\left(\frac{1+R^2}{1-R^2}\right)\tan\theta}{a+d\left(\frac{1+R^2}{1-R^2}\right)\tan\theta} \text{ and we can choose}$$

$$A = \begin{pmatrix} 1 & \frac{R^2-1}{R^2+1}\tan\phi \\ \frac{1+R^2}{1-R^2}\tan\phi & 1 \end{pmatrix}.$$

When we take the determinant $\delta = \tan^2\phi + 1 = \sec^2\phi$, and trace $\tau = 2$, then

$$\begin{aligned} \tau^2 - 4\delta &= 4 - 4\sec^2\phi &< 0 \\ &= 4\left(1 - \sec^2\phi\right) &< 0, \text{ if } \phi \neq 0. \end{aligned}$$

Thus, the resulting dilated conic determined by the locus of intersection is always an ellipse.

Note that if transformation T is an opposite isometry then the image of a line through $-R$ is obtained by complex conjugation followed by a direct transformation. Since the Hermitian matrix of the complex conjugate of a line is represented by the transpose of the Hermitian matrix of the line we only need to replace θ with $-\theta$ to obtain the Hermitian matrix of the image line, whereby the slope of the chord after dilation is $M = \frac{1+R^2}{1-R^2}\tan(\phi - \theta)$. Thus we find the affine matrix A from the equation

$$\begin{aligned} \frac{1+R^2}{1-R^2}\tan(\phi-\theta) &= \frac{1+R^2}{1-R^2}\frac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta} \\ &= \frac{c + d\frac{1+R^2}{1-R^2}\tan\theta}{a + b\frac{1+R^2}{1-R^2}\tan\theta}, \end{aligned}$$

this yields

$$A = \begin{pmatrix} 1 & \frac{1-R^2}{1+R^2}\tan\phi \\ \frac{1+R^2}{1-R^2}\tan\phi & -1 \end{pmatrix}.$$

The determinant $\delta = -\tan^2\phi - 1 = -\sec^2\phi$, and the trace $\tau = 0$, so $\tau^2 - 4\delta = 4\sec^2\phi > 0$. Thus the opposite case of the dilated conic determined by the locus of intersection always results in a hyperbola. So with $r = \frac{2R}{1+R^2}$ and a, b, c, and d from above,

$$cx^2 + (d-a)\,xy - by^2 - (a+d)\,ry - cr^2 = 0$$

becomes

$$\left(\frac{R^2+1}{1-R^2}\right)x^2 - 2xy\cot\phi - \left(\frac{1-R^2}{R^2+1}\right)y^2 - \frac{4R^2}{(R^2+1)(1-R^2)} = 0$$

which simplifies to

$$x^2\cosh d - 2xy\cot\phi - y^2\frac{1}{\cosh d} = (\tanh d)(\sinh d).$$

Since this hyperbola is symmetric about the center of the disk it can always be rotated about the origin into standard position and there are four distinct intersections with the boundary circle $\mathfrak{C}$. This yields the following opposite case.

Opposite Case

If we let $R = \frac{1}{2}$ and $\phi = \frac{\pi}{3}$, then

$$\left(\frac{R^2+1}{1-R^2}\right)x^2 - 2xy\cot\phi - \left(\frac{1-R^2}{R^2+1}\right)y^2 - \frac{4R^2}{(R^2+1)(1-R^2)} = 0$$

simplifies to

$$25x^2 - 10\sqrt{3}xy - 9y^2 - 16 = 0. \text{ See Figure 4.1.}$$

We can contract the equation of the hyperbolic conic by substituting $u = \frac{2x}{x^2+y^2+1}$, $v = \frac{2y}{x^2+y^2+1}$, and $d = ln\frac{1+R}{1-R}$ into the equation

$$x^2\cosh d - 2xy\cot\phi - y^2\frac{1}{\cosh d} = (\tanh d)(\sinh d)$$

which results in the following equation

$$\left(\frac{4\cosh(ln3)}{(x^2+y^2+1)^2}\right)x^2 - \left(\frac{8\sqrt{3}}{3(x^2+y^2+1)^2}\right)xy - \left(\frac{4}{(\cosh(ln3))(x^2+y^2+1)^2}\right)y^2 - \tan(ln3)\sin(ln3) = 0.$$

Figure 4.2 is the graph of the equation above combined with the graph of the hyperbola created by the opposite case as in Figure 4.1.

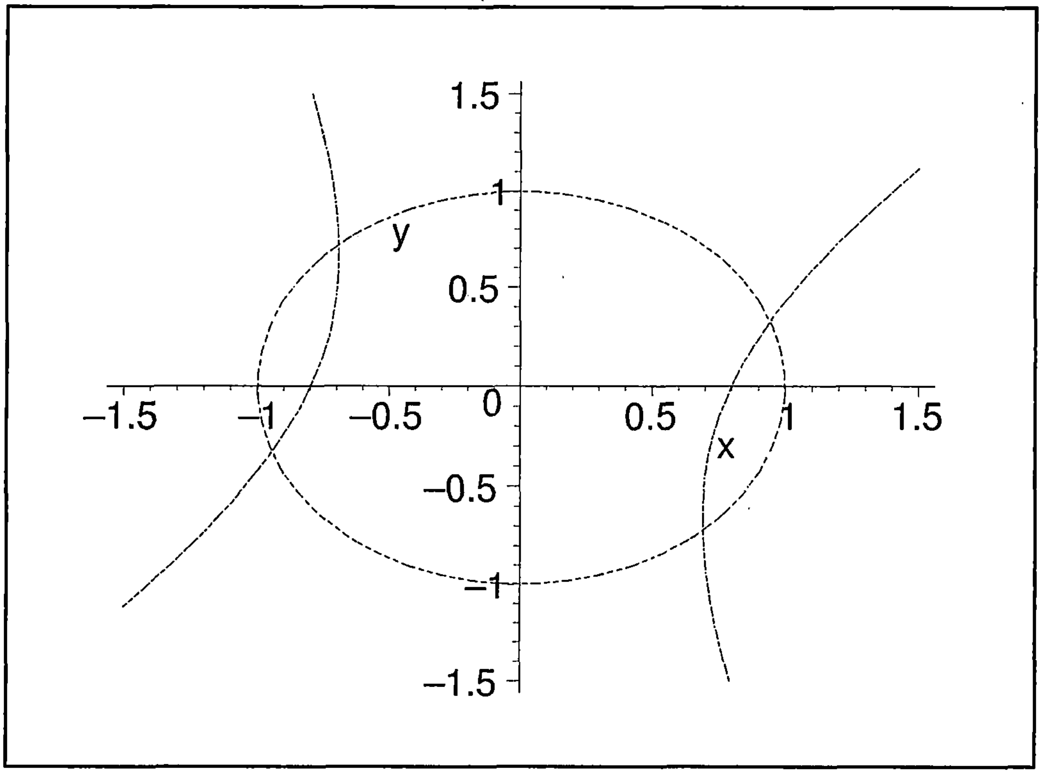

Figure 4.1: A hyperbola created by the opposite case.

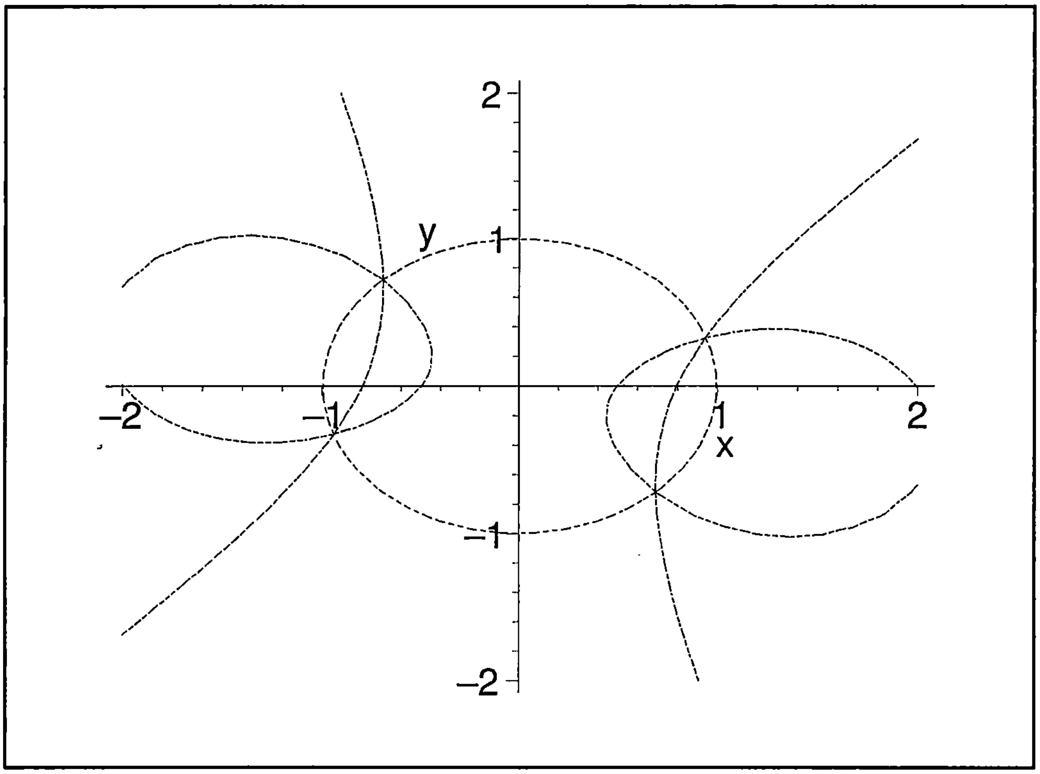

Figure 4.2: The contracted conic.

Chapter 5

Classification of Hyperbolic Conics

The previous work shows that there are potentially six types of hyperbolic conics (as opposed to the three types of affine Euclidean conic). Recall in the Euclidean affine plane we consider three cases:

1) Ellipse, if the line at infinity is an exterior line,
2) Parabola, if the line at infinity is a tangent,
3) Hyperbola, if the line at infinity is a secant.

In the hyperbolic plane the line at infinity is replaced by the unit circle $\mathfrak{C}$. So the type of conic is determined by the way this circle intersects the locus. Since these intersections are the same for the dilated conic we only need to consider the number of ways an ordinary Euclidean conic can intersect the unit circle.

Let's consider our dilated ellipse in standard form

$$\frac{(1+R^2)^2}{4R^2\csc^2\phi}x^2 + \frac{(1-R^2)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1.$$

For $0 < \phi < \pi$ we have $\csc\phi > 0$, so the vertices of the ellipse on the major axis (imaginary axis) are located at $\frac{2R}{1-R^2}\cot\phi \pm \frac{2R}{1-R^2}\csc\phi$. In particular, the upper vertex is on the positive axis and the lower vertex is on the negative axis. This leads to the following six cases:

Case 1: No Intersection

If $\frac{2R}{1-R^2}\cot\frac{\phi}{2} < 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} < 1$ and we let $R = \frac{1}{4}$ and $\phi = \frac{\pi}{3}$, then

$\frac{(1+R^2)^2}{4R^2\csc^2\phi}x^2 + \frac{(21-R^2)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes $\frac{867}{256}x^2 + \frac{675}{256}\left(y - \frac{8}{45}\sqrt{3}\right)^2 = 1$.
See Figure 5.1.

Case 2: Unique Intersection; A Single Tangent

If $\frac{2R}{1-R^2}\cot\frac{\phi}{2} < 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} = 1$, or $\frac{2R}{1-R^2}\cot\frac{\phi}{2} = 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} < 1$ and we

let $R = 2-\sqrt{3}$ and $\phi = \frac{\pi}{3}$, then$\frac{(1+R^2)^2}{4R^2\csc^2\phi}x^2 + \frac{(1-R^2)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes

$3x^2 + \frac{9}{4}\left(y - \frac{1}{3}\right)^2 = 1$. See Figure 5.2.

Case 3: Two Intersections; Both Tangents

If we $\frac{2R}{1-R^2}\cot\frac{\phi}{2} = 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} = 1$. For example, let $R = \sqrt{2}-1$ and $\phi = \frac{\pi}{2}$, then

$\frac{(1+R^2)^2}{4R^2\csc^2\phi}x^2 + \frac{(1-R^2)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes $2x^2 + y^2 = 1$. See Figure 5.3.

Case 4: Two Intersections, A Secant

If $\frac{2R}{1-R^2}\cot\frac{\phi}{2} < 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} > 1$, or $\frac{2R}{1-R^2}\cot\frac{\phi}{2} > 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} < 1$ and we

let $R = \frac{1}{2}$ and $\phi = \frac{\pi}{3}$, then $\frac{(1+R^2)^2}{4R^2\csc^2\phi}x^2 + \frac{(1-R^2)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes

$\frac{75}{64}x^2 + \frac{27}{64}\left(y - \frac{4}{9}\sqrt{3}\right)^2 = 1$. See Figure 5.4.

<u>Case 5</u>: Three intersections, A Tangent and A Secant

If $\frac{2R}{1-R^2}\cot\frac{\phi}{2} > 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} = 1$, or $\frac{2R}{1-R^2}\cot\frac{\phi}{2} = 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} > 1$ and we let $R = \frac{1}{\sqrt{3}}$ and $\phi = \frac{\pi}{3}$, then $\frac{\left(1+R^2\right)^2}{4R^2\csc^2\phi}x^2 + \frac{\left(1-R^2\right)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes $x^2 + \frac{1}{4}\left(y-1\right)^2 = 1$. See Figure 5.5.

<u>Case 6</u>: Four Intersections, Two Secants

If $\frac{2R}{1-R^2}\cot\frac{\phi}{2} > 1$ and $\frac{2R}{1-R^2}\tan\frac{\phi}{2} > 1$ and we let $R = \frac{1}{\sqrt{3}}$ and $\phi = \frac{\pi}{3}$, then

$\frac{\left(1+R^2\right)^2}{4R^2\csc^2\phi}x^2 + \frac{\left(1-R^2\right)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$ becomes $\frac{169}{192}x^2 + \frac{25}{192}\left(y - \frac{4}{5}\sqrt{3}\right)^2 = 1$.
See Figure 5.6.

Note: The equation of the ellipse can be written in terms of distance between the two original points in the hyperbolic plane, d; and the angle of rotation ϕ that determines T. The equation of our dilated ellipse becomes

$$\frac{1}{\left((\tanh d)(\csc\phi)\right)^2}x^2 + \frac{1}{\left((\sinh d)(\csc\phi)\right)^2}\left(y - (\sinh d)\,(\cot\phi)\right)^2 = 1,$$

where $R = \tanh\frac{d}{2}$ and $\frac{2R}{1-R^2} = 2\frac{\tanh\frac{d}{2}}{1-\tanh\frac{d}{2}} = \sinh d$.

To find the actual hyperbolic conic suppose that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ produces the Cartesian conic $\frac{\left(1+R^2\right)^2}{4R^2\csc\phi}x^2 + \frac{\left(1-R^2\right)^2}{4R^2\csc^2\phi}\left(y - \frac{2R\cot\phi}{1-R^2}\right)^2 = 1$.

Since this conic is the result of dilating the hyperbolic conic, we can obtain the equation of the hyperbolic conic by the usual inverse transformation procedure, that is, if (x, y) is a point on the hyperbolic conic then the dilation of (x, y) must satisfy the above quadratic. The dilation of (x, y) is $\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}\right)$. So, if we let $u = \frac{2x}{1+x^2+y^2}$ and $v = \frac{2y}{1+x^2+y^2}$ then the above equation results in the following

$$\frac{\left(1+R^2\right)^2}{4R^2\csc^2\theta}u^2 + \frac{\left(1-R^2\right)^2}{4R^2\csc^2\phi}\left(v - \frac{2R\cot\phi}{1-R^2}\right)^2 - 1 = 0$$

and becomes,

$$\frac{1}{R^2}x^2\left(\sin^2\phi\right)\frac{\left(R^2+1\right)^2}{(x^2+y^2+1)^2}+\frac{1}{4R^2}\left(\sin^2\phi\right)\left(1-R^2\right)^2\left(2\frac{y}{r^2+1}-2R\frac{\cot\phi}{1-R^2}\right)^2-1=0$$

which simplifies to the biquadratic or quartic form

$$r^4+\left(r^2+1\right)y\tfrac{2}{R}\left(\cot\phi\right)\left(1-R^2\right)-r^2\left(R^4+1\right)+4y^2+1=0$$

where $r^2=x^2+y^2$ and $R=\tan\frac{d}{2}$.

If we use polar coordinates, $x=r\cos\theta$ and $y=r\sin\theta$, the above equation simplifies into $r^4+Ar^3+Br^2+Ar+1=0$.

This is the polar equation of the contraction to the hyperbolic conic, where

$$\begin{aligned}
A &= 2\frac{\left(1-R^2\right)\cot\phi}{R}\sin\theta &&= \frac{4\cot\phi}{\sinh d}\sin\theta \\
&= 4\left(\cot\phi\right)\left(\operatorname{csch}d\right)\sin\theta \\
B &= -\frac{1}{R^2}\left(R^4-4R^2\sin^2\theta+1\right) &&= -\left(\tanh^2 p-4\sin^2\theta+\coth^2 p\right) \\
&= 4\sin^2\theta-2\frac{1+\cosh^2 d}{\sinh^2 d} &&= -2\left(\cos 2\theta+2\operatorname{csch}^2 d\right)
\end{aligned}$$

The symmetry in polar coefficients results from the fact tht the curve must be invariant under the interchange of r with $\frac{1}{r}$, meaning that the curve is invariant under inversion in the unit circle. The portion of the curve outside the unit circle results from the locus of second intersections of the circular arcs through $-R$ and R. We will now use the contracted conic above in each of the previous six cases.

Case 1: No Intersection

Let $A = 2\frac{(1-R^2)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = \frac{1}{4}$, and $\phi = \frac{\pi}{3}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes

$r^4 + \frac{5}{2}\sqrt{3}\,(\sin\theta)\,r^3 + \left(4\sin^2\theta - \frac{257}{16}\right)r^2 + \frac{5}{2}\sqrt{3}\,(\sin\theta)\,r + 1 = 0$. See Figure 5.7.

Case 2: A Unique Intersection; A Single Tangent

Let $A = 2\frac{(1-R^2)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = 2 - \sqrt{3}$, and $\phi = \frac{\pi}{3}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes

$r^4 + 4\,(\sin\theta)\,r^3 + \left(4\sin^2\theta - 14\right)r^2 + 4\,(\sin\theta)\,r + 1 = 0$. See Figure 5.8.

Case 3: Two Intersections; Both Tangents

Let $A = 2\frac{(1-R^2)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = \sqrt{2} - 1$, and $\phi = \frac{\pi}{2}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes $r^4 + \left(4\sin^2\theta - 6\right)r^2 + 1 = 0$.
See Figure 5.9. Notice that if we change to Cartesian coordinates this graph is a picture of two circles intersecting on the boundary and has Cartesian factors of
$\left(x^2 - 2x + y^2 - 1\right)\left(x^2 + 2x + y^2 - 1\right) = 0$.

Case 4: Two Intersection; A Secant

Let $A = 2\frac{(1-R^2)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = \frac{1}{2}$, and $\phi = \frac{\pi}{3}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes

$r^4 + \sqrt{3}\,(\sin\theta)\,r^3 + \left(4\sin^2\theta - \frac{17}{4}\right)r^2 + \sqrt{3}\,(\sin\theta)\,r + 1 = 0$. See Figure 5.10.

Case 5: Three Intersections; A Secant and A Tangent

Let $A = 2\frac{\left(1-R^2\right)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = \frac{\sqrt{3}}{3}$, and $\phi = \frac{\pi}{3}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes

$r^4 + \frac{4}{3}\left(\sin\theta\right)r^3 + \left(4\sin^2\theta - \frac{10}{3}\right)r^2 + \frac{4}{3}\left(\sin\theta\right)r + 1 = 0$. See Figure 5.11.

Case 6: Four Intersections; Two Secants

Let $A = 2\frac{\left(1-R^2\right)\cot\phi}{R}\sin\theta$, $B = -\frac{1}{R^2}\left(R^4 - 4R^2\sin^2\theta + 1\right)$, $R = \frac{2}{3}$, and $\phi = \frac{\pi}{3}$,

then $r^4 + Ar^3 + Br^2 + Ar + 1 = 0$ becomes

$r^4 + \frac{5\sqrt{3}}{9}\left(\sin\theta\right)r^3 + \left(4\sin^2\theta - \frac{97}{36}\right)r^2 + \frac{5\sqrt{3}}{9}\left(\sin\theta\right)r + 1 = 0$. See Figure 5.12.

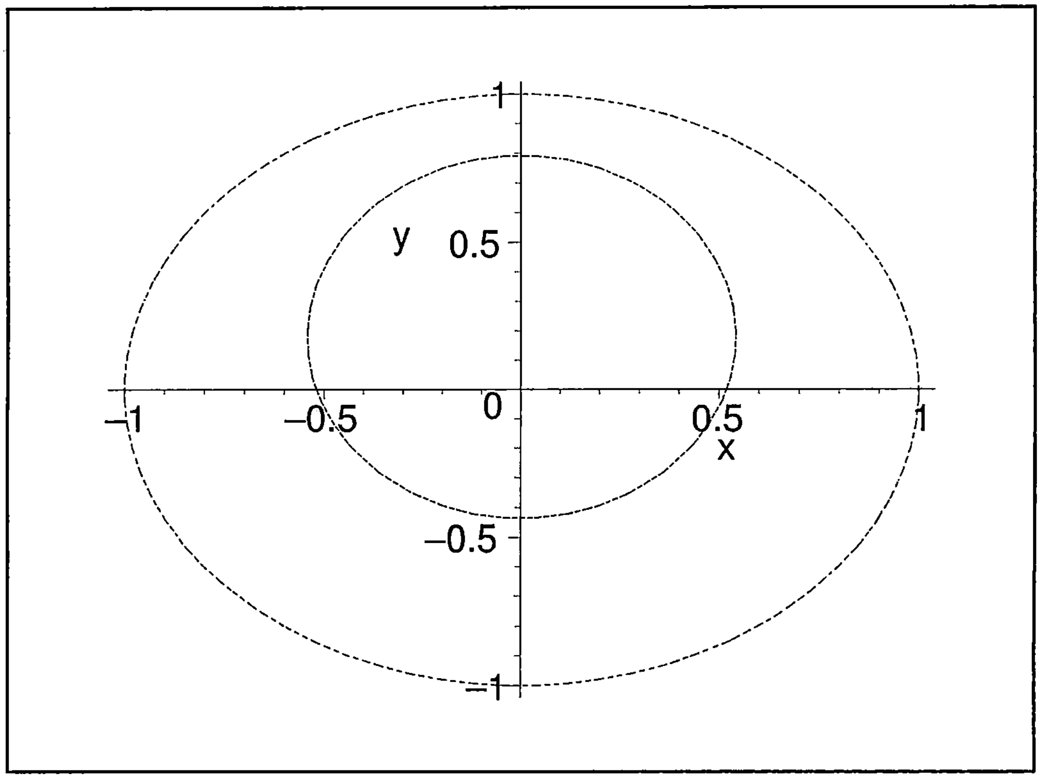

Figure 5.1: The ellipse with no intersections.

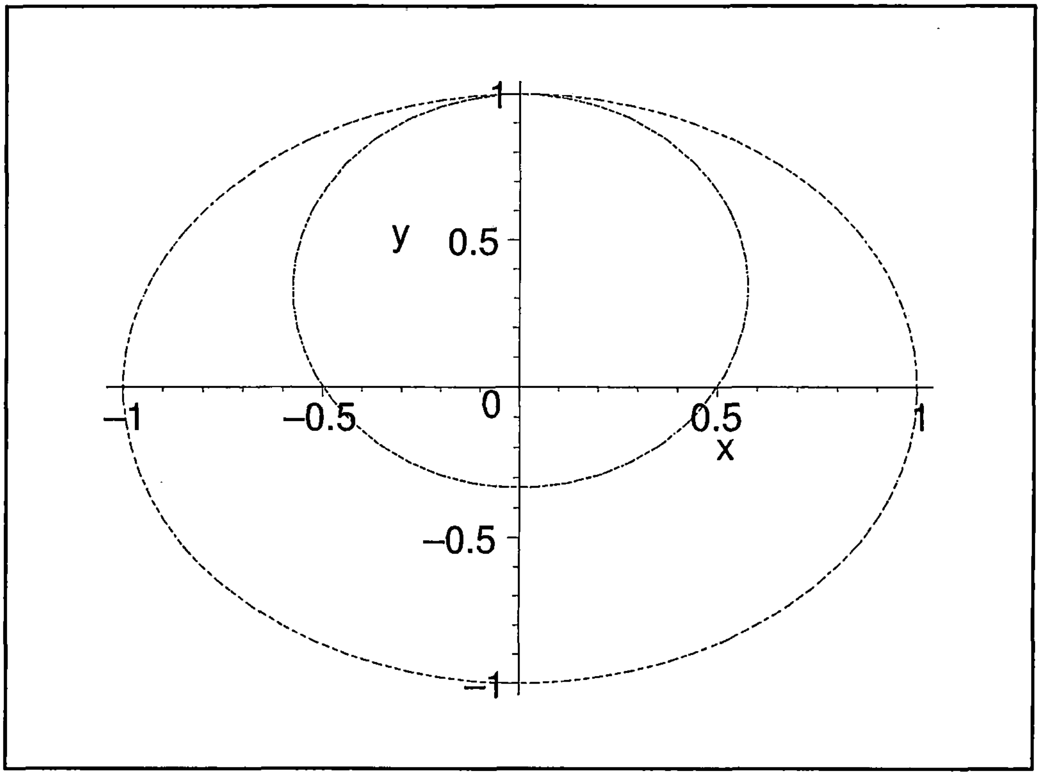

Figure 5.2: The ellipse with a unigue intersection, a single tangent.

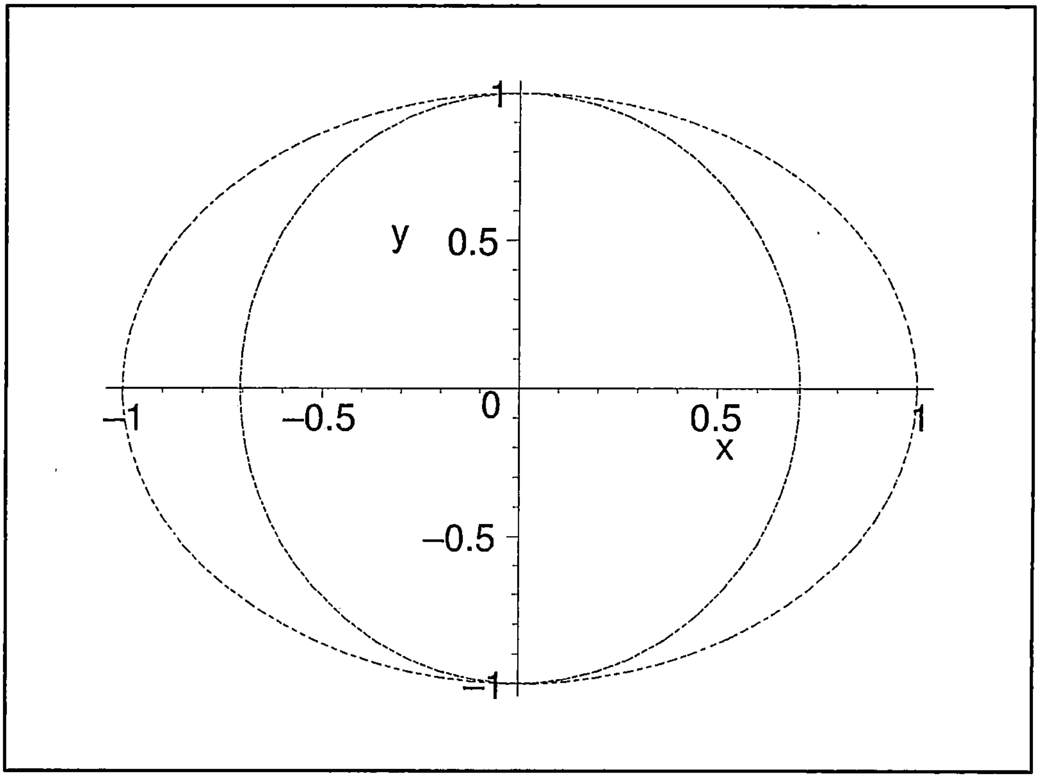

Figure 5.3: The ellipse with two intersections, both tangents.

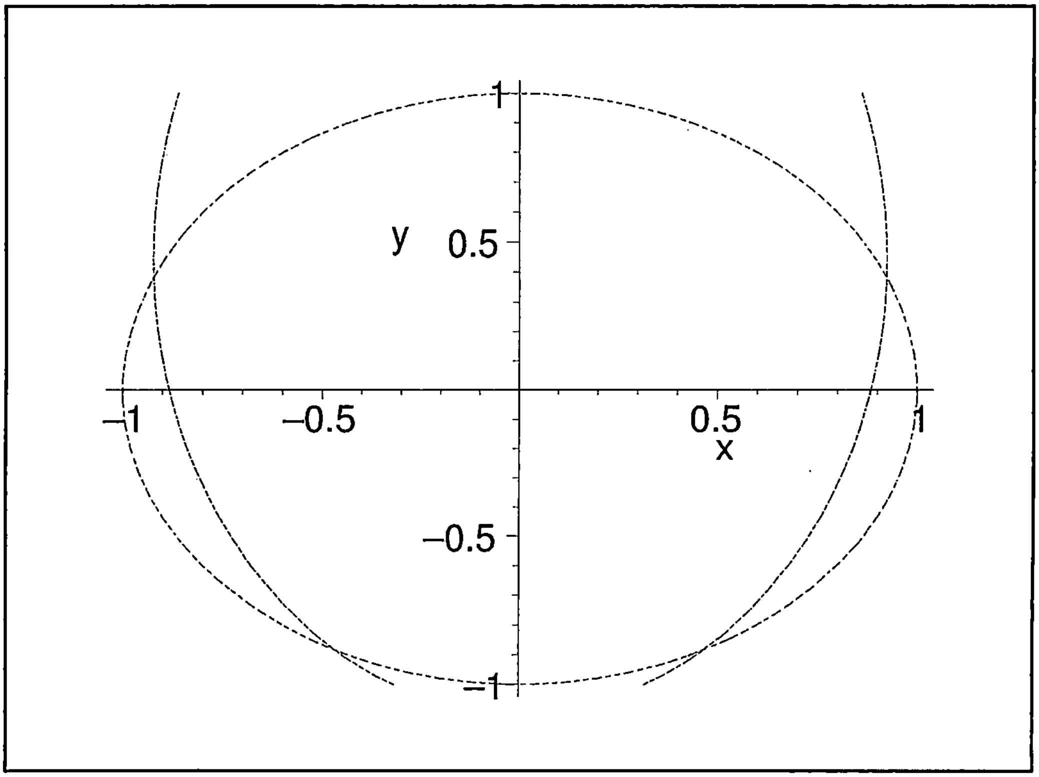

Figure 5.4: The ellipse with two intersections, a secant.

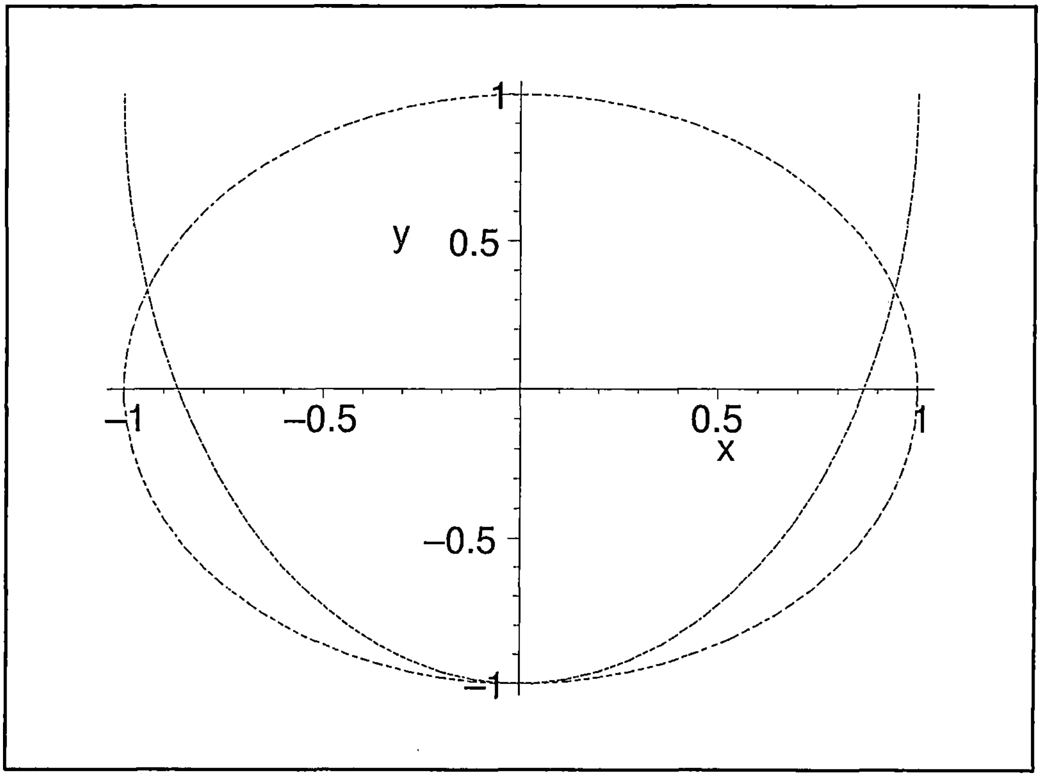

Figure 5.5: The ellipse with two intersections, a tangent and a secant.

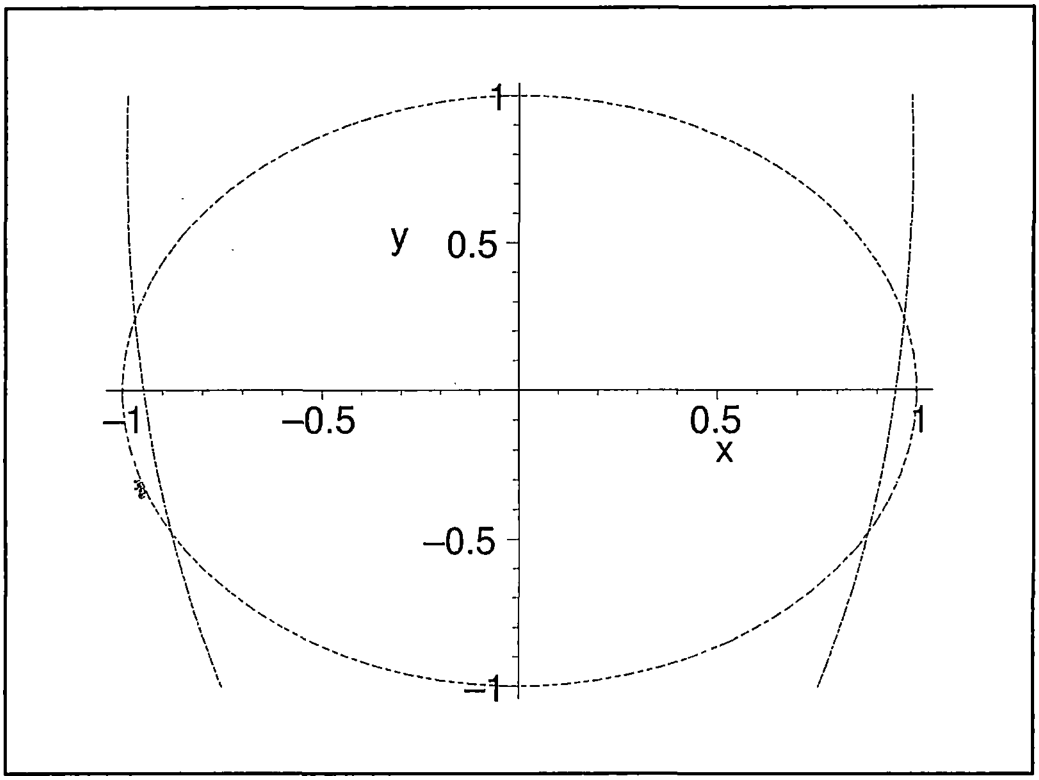

Figure 5.6: The ellipse with two intersections, two secants.

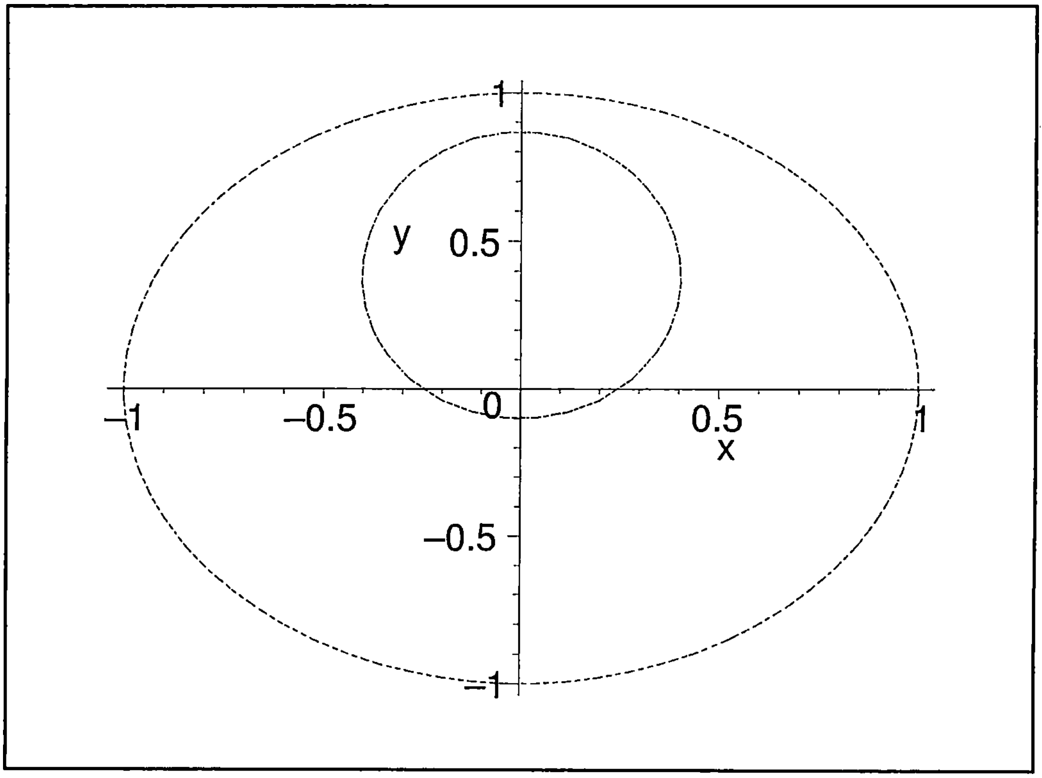

Figure 5.7: The ellipse with no intersections.

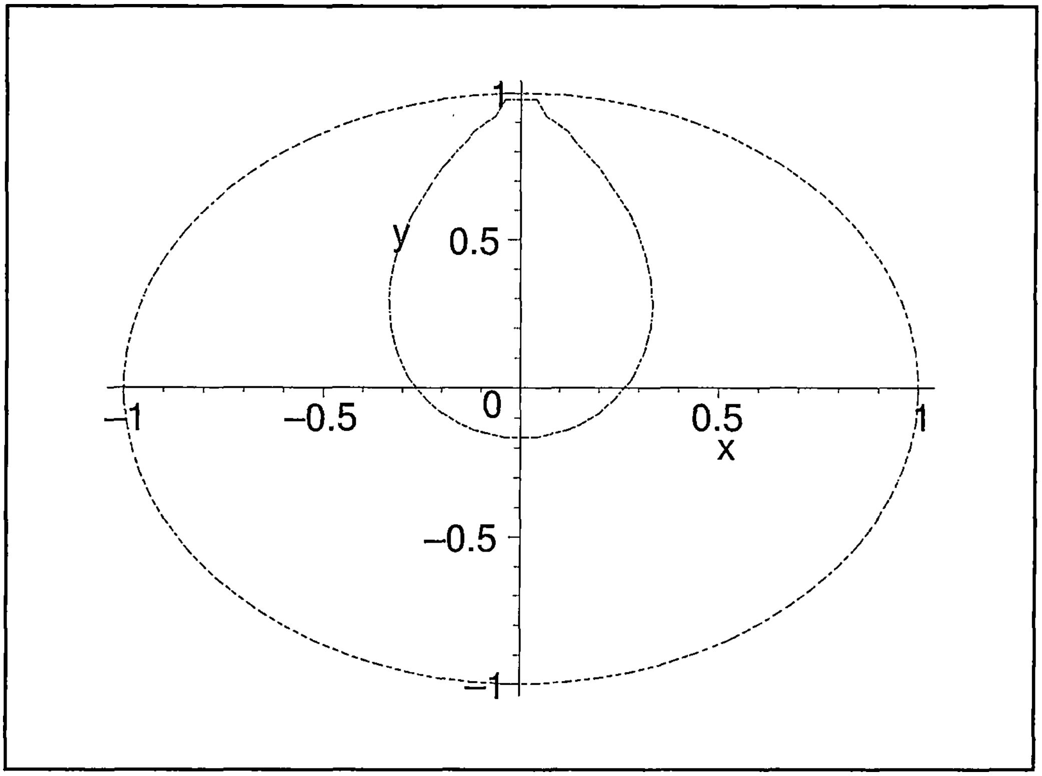

Figure 5.8: The ellipse with one unique intersection.

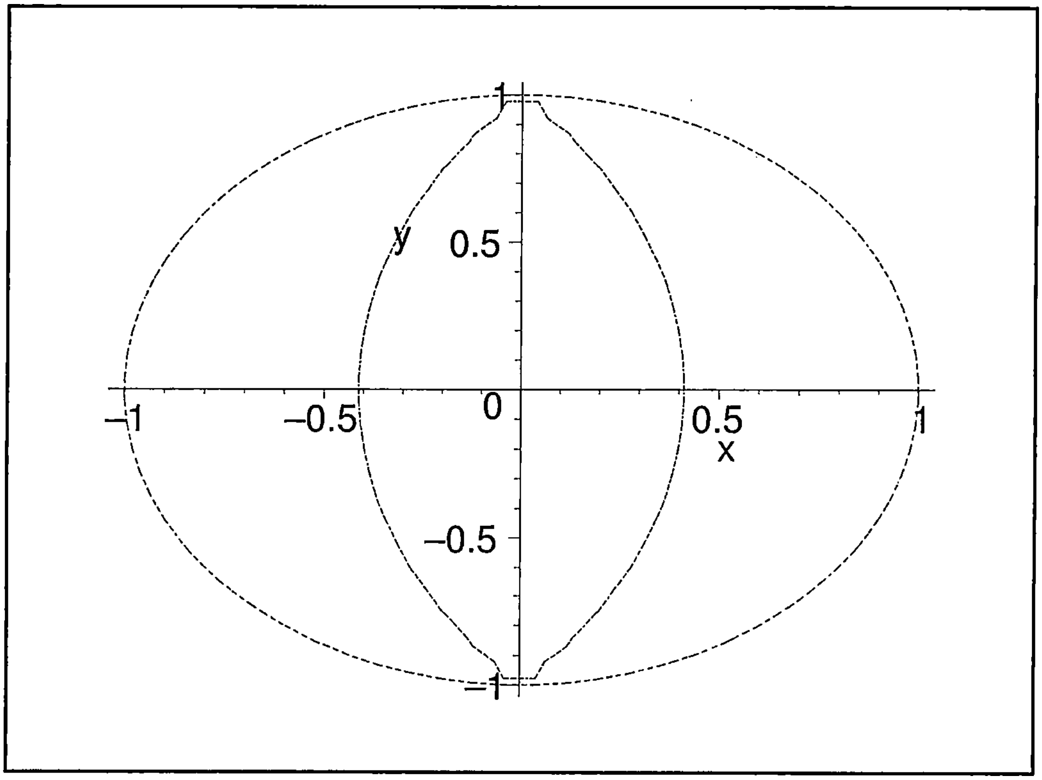

Figure 5.9: The ellipse with two intersections as tangents.

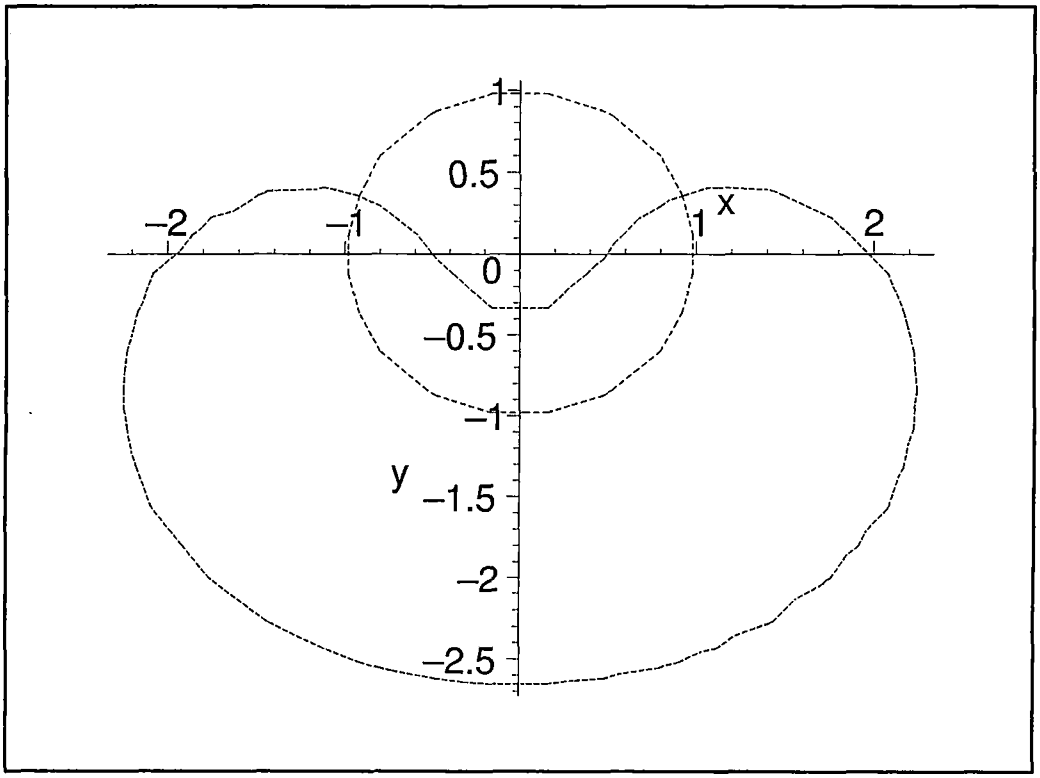

Figure 5.10: The ellipse with two intersections as a secant.

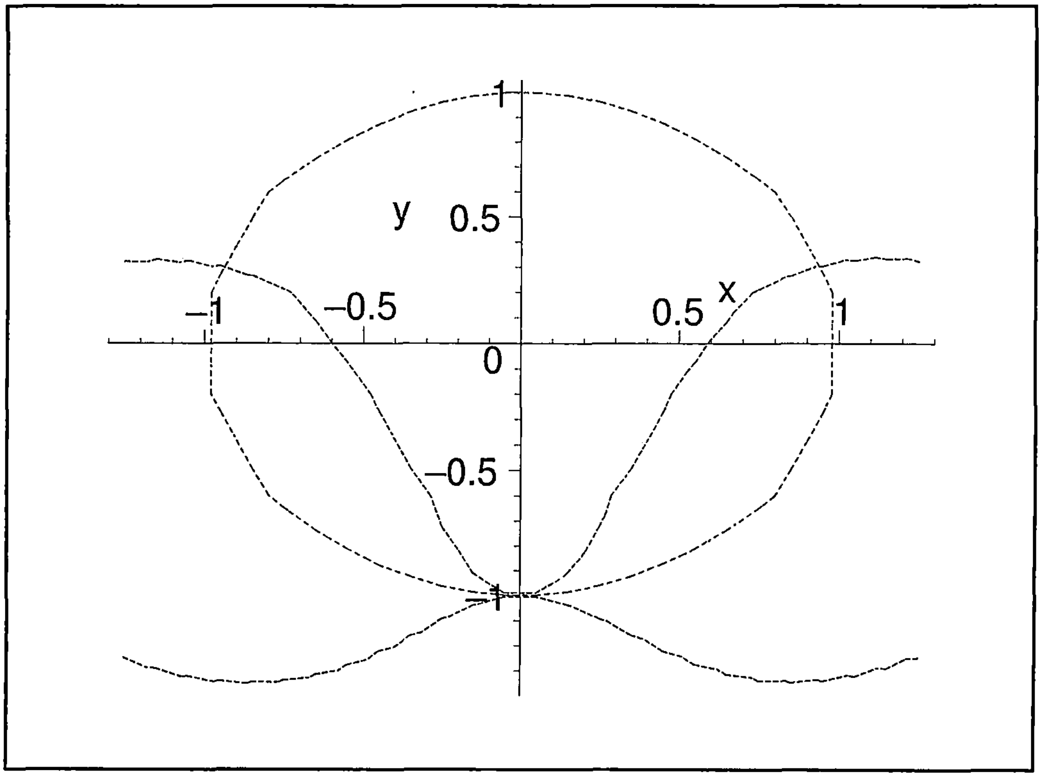

Figure 5.11: The ellipse with three intersections.

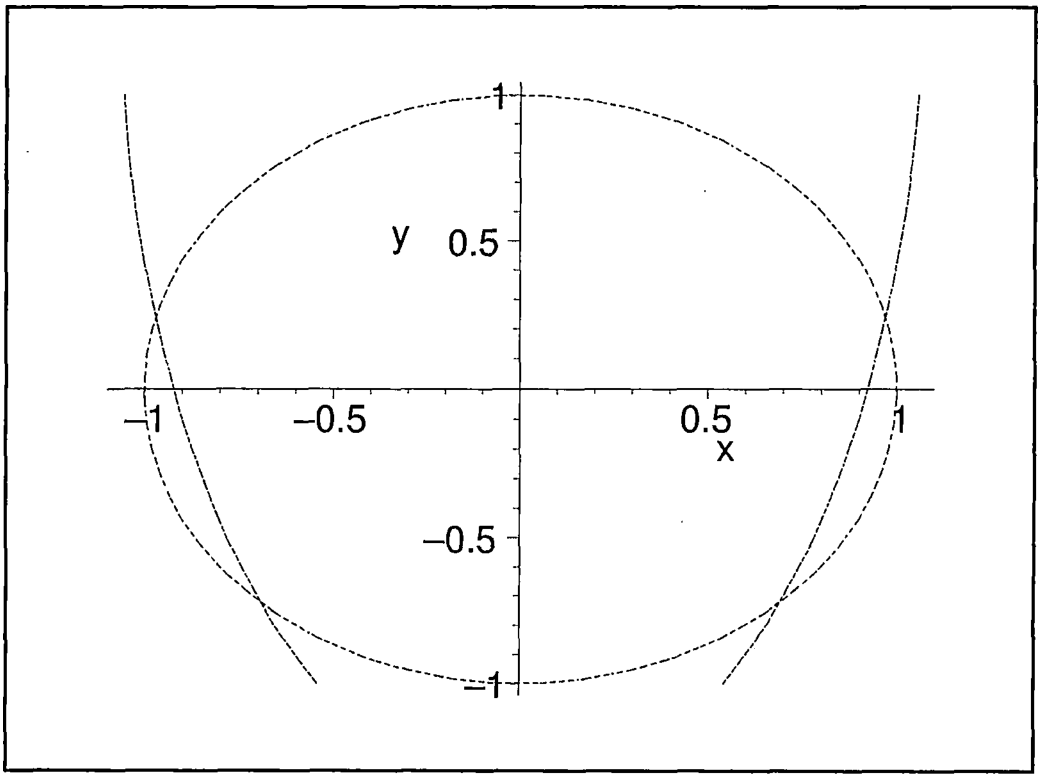

Figure 5.12: The ellipse with four intersections.

Chapter 6

Conclusion

In this body of work, we have studied and shown how conics arise in planar geometry as a linear correspondence between two concurrent pencils. The locus of intersections $L \cap T(L)$, as L runs through the entire pencil of lines through a point P, and where T is an affine transformation such that $T(P) = Q$, creates an affine conic: an ellipse, a parabola, or a hyperbola.

When we determined the intersection of L with $T(L)$ we found the following equation of the affine conic,

$$cx^2 + (d-a)\,xy - by^2 - (a+d)\,ry - cr^2 = 0.$$

The type of conic was determined by the invariants of T, the determinant and trace of the linear parts of the matrix T. This led to three cases of the affine conics.

We used this affine result to explore conics in the hyperbolic plane using Poincaré's conformal disk model, where the lines of the geometry are segments of circles contained in the disk orthogonal to the boundary circle $\mathfrak{C}$. We then chose two points P and Q and a transformation T which takes the pencil of circles through P to the pencil through Q. The transformation T is a linear fractional transformation that preserves the disk.

We finally showed that the conic produced by T is mapped to an affine conic when dilation about the center of the disk by a hyperbolic factor of two is imposed. This dilation takes the arc to chords of the disk that share the same boundary points and gives rise to a map, called the spinor map, that allows us to classify the hyperbolic conic by examining it's affine counterpart. Since the affine conic is always an ellipse, we were able to classify the hyperbolic conic by counting intersections of this affine conic with the boundary of the disk. This resulted in six cases and the actual hyperbolic conic can be recovered in explicit form by contracting the affine conic.

Bibliography

[BEG99] David A. Brannan, Matthew F. Esplen, and Jeremy J. Gray. *Geometry*. Cambridge University Press, Cambridge, 1999.

[Ped88] Dan Pedoe. *Geometry*. Dover Books on Advanced Mathematics. Dover Publications Inc., New York, second edition, 1988. A comprehensive course.

[Sar03] John Sarli. Polar inversion and conics. pages 1–17, 2003. Unpublished Expository Work.

[Sch79] Hans Schwerdtfeger. *Geometry of complex numbers*. Dover Publications Inc., New York, 1979. Circle geometry, Moebius transformation, non-Euclidean geometry, A corrected reprinting of the 1962 edition, Dover Books on Advanced Mathematics.